

The exact quantum value of a cyclic Bell operator

A sharp polar-decomposition certificate for all dimensions

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with heavy assistance from ChatGPT 5.6 Sol

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Abstract

Perito, D’Avino, Jung, Mironowicz, Acin, and Augusiak recently introduced a cyclic family of Bell operators and conjectured that its Tsirelson bound is $2 \csc(\pi/(2d))$ for every integer $d \geq 2$. We prove the conjecture. In fact, the operator upper bound holds after relaxing every observable to an arbitrary unitary and even in the commuting-operator model; the order- d relations and tensor-product representation are not needed for the upper bound. The proof combines an exact positive-factor identity from polar decomposition with the sharp scalar formula

$$\max_{|z|=1} \sum_{y=0}^{d-1} |1 + e^{2\pi i y/d} z| = 2 \csc\left(\frac{\pi}{2d}\right).$$

We characterize equality in the scalar formula, give an explicit functional-calculus certificate for the operator inequality, and include a self-contained verification that the known d -dimensional strategy is admissible and saturates the bound. This determines the quantum value only; it does not establish uniqueness, self-testing, or all-dimensional randomness certification.

Status. This is an unrefereed, AI-assisted research note. The analytic argument is accompanied by symbolic and numerical checks, but has not received independent expert review.

1 Introduction

Let $d \geq 2$ and $\omega = e^{2\pi i/d}$. For unitary observables A_0, A_1 for Alice and $B_0, \dots, B_{d-1}$ for Bob, Perito et al. [1] consider

$$\mathcal{I}_d = \frac{1}{2} \sum_{x=0}^1 \sum_{y=0}^{d-1} \omega^{xy} A_x \otimes B_y + \text{h.c.} = \sum_{y=0}^{d-1} \text{Re}((A_0 + \omega^y A_1) \otimes B_y). \quad (1)$$

In the Bell scenario the observables obey $A_x^d = B_y^d = \mathbf{1}$. The originating paper constructs a d -dimensional strategy with value

$$\tau_d = 2 \csc\left(\frac{\pi}{2d}\right), \quad (2)$$

and conjectures that this is optimal. It proves the weaker general upper bound $d\sqrt{2}$ in Appendix A.2, Proposition 3, and reports numerical tightness of (2) through $d = 6$ using the NPA hierarchy [10, 11].

The $d = 3$ member has the structure of the Buhrman–Massar functional [2] with one Alice setting removed [1]. Nearby exact-bound constructions include the SATWAP family tailored to maximally entangled states [3], a modified Buhrman–Massar family with an SOS certificate and mutually unbiased bases [4], and a recent exact certificate and robust self-test for the original

CHSH mod 3 functional [5]. Recent contemporaneous work on global randomness [6, 7] and a general SOS framework for high-dimensional Bell inequalities [8] does not supply the bound proved here.

Our contribution is the missing analytic upper bound. The essential point is that the Bob operators can be optimized term by term by polar decomposition without destroying the coupling between A_0 and A_1 . That coupling is then entirely encoded by the relative unitary $A_0^\dagger A_1$, reducing the problem to a scalar trigonometric extremum on the unit circle.

Theorem 1.1 (Sharp commuting-operator bound). *Let $d \geq 2$. On an arbitrary complex Hilbert space, let $A_0, A_1, B_0, \dots, B_{d-1}$ be unitaries such that $[A_x, B_y] = 0$ for every x, y , and interpret $\mathcal{I}_d$ by replacing $A_x \otimes B_y$ in (1) with $A_x B_y$. Then*

$$\mathcal{I}_d \leq 2 \csc\left(\frac{\pi}{2d}\right) \mathbf{1} \quad (3)$$

as an operator inequality. Consequently, with the order- d relations imposed, the finite-dimensional tensor-product, approximate, and commuting-operator values all coincide:

$$\boxed{\beta_q(\mathcal{I}_d) = \beta_{qa}(\mathcal{I}_d) = \beta_{qc}(\mathcal{I}_d) = 2 \csc\left(\frac{\pi}{2d}\right)} \quad (d \geq 2).$$

Here q denotes the finite-dimensional tensor model, qa its closure, and qc the commuting-operator model. We use the standard inclusions

$$\mathcal{Q}_q \subseteq \mathcal{Q}_{qa} \subseteq \mathcal{Q}_{qc}.$$

Together with the finite-dimensional attaining strategy, they give the equality of the three values. Thus both relaxing the order relations and passing to commuting operators leave the optimum unchanged.

2 A polar-decomposition identity

For a bounded operator C , write $|C| = (C^\dagger C)^{1/2}$ and $|C^\dagger| = (C C^\dagger)^{1/2}$.

Lemma 2.1 (Polar positive-factor identity). *Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ be commuting unital $*$ -algebras, let $C \in \mathcal{A}$, let $C = V|C|$ be its polar decomposition, and let $B \in \mathcal{B}$ be unitary. Define*

$$P_{C,B} = |C^\dagger|^{1/2} - V|C|^{1/2}B.$$

Then

$$\frac{|C| + |C^\dagger|}{2} - \operatorname{Re}(CB) = \frac{1}{2} P_{C,B}^\dagger P_{C,B}. \quad (4)$$

Proof. The polar factor V may be a partial isometry. Its initial projection is the support projection of $|C|$, so

$$|C|^{1/2} V^\dagger V |C|^{1/2} = |C|.$$

Polar functional calculus also gives

$$|C^\dagger|^{1/2} V = V|C|^{1/2}, \quad |C^\dagger|^{1/2} V |C|^{1/2} = C.$$

The polar operators belong to the von Neumann algebra generated by $\mathcal{A}$, so they commute with B . Expanding with all factors displayed gives

$$P_{C,B}^\dagger P_{C,B} = |C^\dagger| + B^\dagger |C|^{1/2} V^\dagger V |C|^{1/2} B - CB - B^\dagger C^\dagger = |C| + |C^\dagger| - CB - (CB)^\dagger.$$

Dividing by two proves (4). $\square$

Remark 2.2. No invertibility or finite-dimensional unitary extension of V is needed. This is useful because some of the operators $A_0 + \omega^y A_1$ may be singular for general strategies. The tensor-product identity follows by representing C as $C \otimes \mathbf{1}$ and B as $\mathbf{1} \otimes B$.

3 The scalar roots-of-unity extremum

Lemma 3.1 (Sharp scalar bound and equality set). *For every $d \geq 2$ and every $z \in \mathbb{C}$ with $|z| = 1$,*

$$\sum_{y=0}^{d-1} |1 + \omega^y z| \leq 2 \csc\left(\frac{\pi}{2d}\right). \quad (5)$$

Equality holds if and only if

$$z^d = (-1)^{d-1}. \quad (6)$$

Proof. Write $z = e^{2is}$, put $h = \pi/d$ and $a = h/2 = \pi/(2d)$. Then the left side of (5) is $2G_d(s)$, where

$$G_d(s) = \sum_{y=0}^{d-1} |\cos(s + yh)|.$$

The function G_d has period h .

First suppose $d = 2m + 1$ is odd. We may take $-a \leq s \leq a$, an interval of length $2a = h$, and reindex the summands by $k = -m, \dots, m$. Every angle $s + kh$ lies in $[-\pi/2, \pi/2]$, since $a + mh = \pi/2$, hence

$$G_d(s) = \sum_{k=-m}^m \cos(s + kh) = \frac{\sin(dh/2)}{\sin(h/2)} \cos s = \csc(a) \cos s.$$

The maximum is $\csc(a)$, attained exactly when $s \equiv 0 \pmod{h}$. Equivalently, $z^d = 1 = (-1)^{d-1}$.

Now suppose $d = 2m$ is even. Take $0 \leq s \leq h$. The first m cosines are nonnegative and the last m are nonpositive, so the finite cosine-sum formula gives

$$\begin{aligned} G_d(s) &= \sum_{y=0}^{m-1} \cos(s + yh) - \sum_{y=m}^{2m-1} \cos(s + yh) \\ &= \csc(a) \cos(s - a). \end{aligned}$$

Its maximum is attained exactly at $s \equiv a \pmod{h}$. This is equivalent to $z^d = -1$, again giving (6). Multiplying the maximum of G_d by two proves (5). $\square$

4 Proof of the upper bound

Proof of Theorem 1.1. Set

$$C_y = A_0 + \omega^y A_1, \quad U = A_0^\dagger A_1, \quad M_y = \mathbf{1} + \omega^y U.$$

Then $C_y = A_0 M_y$. The operator M_y is normal, since

$$M_y^\dagger M_y = M_y M_y^\dagger = 2\mathbf{1} + \omega^y U + \omega^{-y} U^\dagger.$$

It follows that

$$|C_y| = |M_y|, \quad |C_y^\dagger| = A_0 |M_y| A_0^\dagger. \quad (7)$$

Define

$$F_d(U) = \sum_{y=0}^{d-1} |\mathbf{1} + \omega^y U|.$$

The continuous functional calculus for the unitary U , together with Lemma 3.1, yields

$$0 \leq F_d(U) \leq \tau_d \mathbf{1}. \quad (8)$$

Apply Lemma 2.1 to C_y and B_y , sum over y , and use (7):

$$\begin{aligned} \mathcal{I}_d &\leq \frac{1}{2} \sum_{y=0}^{d-1} (|C_y| + |C_y^\dagger|) \\ &= \frac{1}{2} (F_d(U) + A_0 F_d(U) A_0^\dagger) \\ &\leq \tau_d \mathbf{1}. \end{aligned}$$

This proves the commuting-operator upper bound. Proposition 5.1 gives a matching finite-dimensional tensor-product strategy. The inclusions of correlation sets stated after Theorem 1.1 complete the proof of all three equalities. $\square$

4.1 An explicit positive-factor certificate

The preceding proof can be packaged as a single exact identity. Let $C_y = V_y |C_y|$ be the polar decomposition, and set

$$\begin{aligned} P_y &= |C_y^\dagger|^{1/2} - V_y |C_y|^{1/2} B_y, \\ G &= (\tau_d \mathbf{1} - F_d(U))^{1/2}. \end{aligned}$$

Then

$$\boxed{\begin{aligned} \tau_d \mathbf{1} - \mathcal{I}_d &= \frac{1}{2} \sum_{y=0}^{d-1} P_y^\dagger P_y \\ &\quad + \frac{1}{2} G^\dagger G \\ &\quad + \frac{1}{2} (G A_0^\dagger)^\dagger (G A_0^\dagger). \end{aligned}} \quad (9)$$

The square roots in (9) are ordinary positive square roots from functional calculus. Thus (9) is an exact operator-level positive-factor certificate, not a finite-level NPA approximation.

5 Tightness and the order- d strategy

The lower bound and the Weyl-based strategy are due to Perito et al. [1]. For completeness, this section records a compact polar form of the same saturating choice and proves directly that its Bob operators satisfy the order relation. This also makes the equality mechanism in Lemma 3.1 transparent.

Proposition 5.1 (Self-contained lower certificate). *On $\mathbb{C}^d$, let*

$$Z|j\rangle = \omega^j|j\rangle, \quad X|j\rangle = |j+1 \bmod d\rangle,$$

and use

$$A_0 = Z, \quad A_1 = X, \quad |\Phi_d\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j, j\rangle.$$

For $0 \leq y < d$, put

$$W_y = \omega^y Z^\dagger X, \quad H_y = |\mathbf{1} + W_y|, \quad (10)$$

$$Q_y = H_y^{-1}(\mathbf{1} + W_y^\dagger)Z^\dagger, \quad B_y = Q_y^T. \quad (11)$$

Then every B_y is unitary, $B_y^d = \mathbf{1}$, and

$$\langle \Phi_d | \mathcal{I}_d | \Phi_d \rangle = \tau_d.$$

Proof. The Weyl relations give $Z^d = X^d = \mathbf{1}$ and $ZX = \omega XZ$. The action of W_y is a weighted cyclic shift:

$$W_y|j\rangle = \omega^{y-j-1}|j+1 \bmod d\rangle.$$

Hence

$$W_y^d = (-1)^{d-1} \mathbf{1}, \quad \text{spec}(W_y) = \{\lambda : \lambda^d = (-1)^{d-1}\}. \quad (12)$$

Indeed, the first d orbit vectors of $|0\rangle$ are nonzero multiples of the standard basis vectors. Thus the minimal polynomial has degree d ; because it divides $t^d - (-1)^{d-1}$, whose roots are distinct, the displayed spectrum is simple and complete. In particular, $-1 \notin \text{spec}(W_y)$, so H_y is invertible.

Let $C_y = Z + \omega^y X = Z(\mathbf{1} + W_y)$. Since W_y is normal,

$$C_y = \underbrace{Z(\mathbf{1} + W_y)H_y^{-1}}_{V_y} H_y$$

is its polar decomposition. Thus $Q_y = V_y^\dagger$, proving that Q_y , and hence B_y , is unitary.

It remains to prove the order relation. On the spectrum in (12), define

$$p(z) = \frac{1+z}{|1+z|}.$$

Then

$$Q_y = p(W_y)^\dagger Z^\dagger;$$

the adjoint is essential because (11) contains $\mathbf{1} + W_y^\dagger$. Since $Z^\dagger W_y Z = \omega^{-1} W_y$,

$$Q_y^d = \prod_{r=0}^{d-1} p(\omega^{-r} W_y)^\dagger. \quad (13)$$

For every eigenvalue λ of W_y , the corresponding scalar in (13) equals

$$\begin{aligned} \prod_{r=0}^{d-1} \overline{p(\omega^{-r} \lambda)} &= \frac{\prod_{r=0}^{d-1} (1 + \omega^{-r} \lambda)}{\prod_{r=0}^{d-1} |1 + \omega^{-r} \lambda|} \\ &= \frac{1 - (-\lambda)^d}{|1 - (-\lambda)^d|} = \frac{2}{2} = 1, \end{aligned}$$

where we used $\prod_{r=0}^{d-1} (1 + a\omega^r) = 1 - (-a)^d$ after noting that $r \mapsto -r$ permutes the residues modulo d , and used $\lambda^d = (-1)^{d-1}$. Therefore $Q_y^d = \mathbf{1}$ and $B_y^d = \mathbf{1}$. In fact, Q_y is a nonzero weighted d -cycle in an eigenbasis of W_y , so its characteristic polynomial is $t^d - 1$; every d -th root occurs.

Finally,

$$\langle \Phi_d | M \otimes N | \Phi_d \rangle = \frac{1}{d} \text{Tr}(MN^T).$$

Since $B_y^T = Q_y = V_y^\dagger$,

$$\langle \Phi_d | C_y \otimes B_y | \Phi_d \rangle = \frac{1}{d} \text{Tr}(C_y V_y^\dagger) = \frac{1}{d} \text{Tr}(H_y),$$

which is real and nonnegative. Multiplication by ω^y permutes the spectrum in (12), so every H_y has the same trace. For any $\lambda_0^d = (-1)^{d-1}$,

$$\text{Tr}(H_0) = \sum_{r=0}^{d-1} |1 + \omega^r \lambda_0| = \tau_d$$

by the equality case of Lemma 3.1. Summing over y therefore gives $d^{-1} \sum_y \text{Tr}(H_y) = \tau_d$, as claimed. $\square$

Remark 5.2. Because C_y is invertible in this construction, equality in the trace polar bound fixes B_y^T to be $V_y^\dagger$. The Bob observables in (11) therefore give a polar-decomposition representation of an attaining strategy. The strategy in Ref. [1] reaches the sum of these termwise trace bounds, so the same equality condition identifies its Bob observables with this polar form. A direct convention check gives

$$\sum_{y=0}^{d-1} B_y = \alpha_d Z^\dagger, \quad \sum_{y=0}^{d-1} \omega^y B_y = \alpha_d X, \quad \alpha_d = \csc\left(\frac{\pi}{2d}\right), \quad (14)$$

which are Eqs. (14)–(17) of Ref. [1]; in particular, $\tau_d = 2\alpha_d$. For $d = 3$, (11) reduces exactly to

$$B_y = \frac{1}{3} (2Z^2 + 2\omega^{2y}X - \omega^{y+1}X^2Z),$$

the explicit formula in that reference, with $\alpha_3 = 2$. The two Fourier constraints in (14) alone leave the remaining Fourier component undetermined; the termwise saturation condition supplies the identification. No new lower-bound strategy is claimed here.

6 Consequences and scope

Corollary 6.1 (The barred functional). *With the convention*

$$\bar{\mathcal{I}}_d = \mathcal{I}_d + \text{Re}(A_0 \otimes B_d),$$

where B_d is a further unitary (and tensor product is replaced by operator product in the commuting model), its three quantum values are

$$\beta_q(\bar{\mathcal{I}}_d) = \beta_{qa}(\bar{\mathcal{I}}_d) = \beta_{qc}(\bar{\mathcal{I}}_d) = 2 \csc\left(\frac{\pi}{2d}\right) + 1.$$

Proof. Theorem 1.1 and $\text{Re}(A_0 B_d) \leq \mathbf{1}$ give the commuting-operator upper bound. Proposition 5.1, augmented by $B_d = Z^\dagger$, saturates the extra term on $|\Phi_d\rangle$. $\square$

This corollary determines a Bell value. It does not prove uniqueness of the maximizing correlation, self-testing, or the maximal-global-randomness claim posed separately in Ref. [1]. Those questions require additional rigidity information not supplied by the value alone.

For the first five dimensions, Theorem 1.1 gives

d	$\beta_q(\mathcal{I}_d)$
2	$2\sqrt{2}$
3	4
4	$2\sqrt{4 + 2\sqrt{2}}$
5	$2(1 + \sqrt{5})$
6	$2(\sqrt{6} + \sqrt{2})$

The $d = 2$ case is the usual CHSH value; the theorem supplies one uniform argument for all d .

7 Verification artifacts

The accompanying source package contains:

- a deterministic verifier for the finite-dimensional matrix identities, including deliberately singular polar factors;
- exact symbolic checks of the displayed low-dimensional radical values and finite trigonometric identities;
- matrix checks of the Weyl spectrum, Bob unitarity and order, saturation, and the operator certificate over a documented range of dimensions;
- direct-sum commuting-operator certificate checks, plus explicit checks of the phase adjoint, both Fourier constraints in (14), and the displayed qutrit formula;
- a machine-readable claim and dependency ledger; and
- SHA-256 hashes for the released sources and rendered paper.

The default strategy sweep covers all $\sum_{d=2}^{12} d = 77$ Bob observables in that range. These checks are secondary validation of the analytic proof. They are not a formal proof assistant verification of the all-dimensional theorem, and the floating-point matrix tests cannot substitute for the arguments above.

Priority, attribution, and disclosure

The Bell family, the conjectured formula, the Weyl construction, the admissible Bob observables, and the matching lower bound are due to Perito, D’Avino, Jung, Mironowicz, Acin, and Augusiak [1]. The narrow contribution claimed here is the analytic upper bound, including its commuting-operator strengthening and the associated exact certificate.

As of 26 July 2026, a focused search found no public proof of Conjecture 1 and no older theorem located in that search that directly supplies this bound. Because the conjecture is recent, that search cannot rule out unindexed, unpublished, or concurrent work. Any priority statement is therefore made only “to the best of our knowledge.”

The proof architecture, derivations, manuscript, verification code, and literature audit were developed with heavy assistance from ChatGPT 5.6 Sol under Alec Kriebel’s direction. No originating author or other researcher was contacted. The work is public to enable checking and remains unrefereed.

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