A reciprocal-square obstruction for equal-sum–product matrices

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Abstract

There is no matrix of globally distinct positive integers with at least three rows whose row sums and column products are all the same number. An elementary argument reduces any number of rows to three selected rows and yields $3 < 2$. The proof is uniform in the dimensions and the common value. It uses only a sum-of-squares identity and a finite telescoping bound. This establishes the conjecture attributed to Joseph DeVincentis on Erich Friedman's November 2002 Math Magic page [1].

1 The unrestricted nonexistence theorem

Theorem 1. *There do not exist integers $m \geq 3$, $n \geq 2$, $N \geq 1$, and an $m \times n$ matrix $A = (a_{ij})$ of pairwise distinct positive integers such that*

$$\sum_{j=1}^{n} a_{ij} = N \quad (1 \leq i \leq m), \qquad \prod_{i=1}^{m} a_{ij} = N \quad (1 \leq j \leq n).$$

Proof. Suppose such a matrix exists. For each column put $x_j = a_{1j}$, $y_j = a_{2j}$, and $z_j = a_{3j}$. All entries outside these three rows are at least 1, so

$$N = x_j y_j z_j \prod_{i=4}^{m} a_{ij} \;\geq\; x_j y_j z_j.$$

For $m = 3$ the displayed extra product is the empty product, equal to 1. Consequently,

$$\begin{aligned} 3 &= \sum_{j=1}^{n} \frac{x_j + y_j + z_j}{N} \\ &\leq \sum_{j=1}^{n} \left(\frac{1}{x_j y_j} + \frac{1}{x_j z_j} + \frac{1}{y_j z_j} \right) \leq \sum_{j=1}^{n} \left(\frac{1}{x_j^2} + \frac{1}{y_j^2} + \frac{1}{z_j^2} \right). \end{aligned} \tag{1}$$

The last inequality follows from the exact identity

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} - \frac{1}{xy} - \frac{1}{xz} - \frac{1}{yz} = \frac{1}{2} \left[\left(\frac{1}{x} - \frac{1}{y} \right)^2 + \left(\frac{1}{x} - \frac{1}{z} \right)^2 + \left(\frac{1}{y} - \frac{1}{z} \right)^2 \right] \geq 0.$$

The $3n$ selected entries are distinct positive integers. If M is their maximum, the last expression in (1) is at most

$$\sum_{k=1}^{M} \frac{1}{k^2} \leq 1 + \sum_{k=2}^{M} \frac{1}{k(k-1)} = 1 + \sum_{k=2}^{M} \left(\frac{1}{k-1} - \frac{1}{k} \right) = 2 - \frac{1}{M} < 2.$$

Thus $3 < 2$, a contradiction. □

2 A stronger inequality, and the boundary cases

The following result concerns ordinary sum–product matrices, for which the common sum S and the common product P need not coincide.

Proposition 2. *Let $m \geq 3$ and $n \geq 1$. Let A have globally distinct positive integer entries, common row sum S, and common column product P. Then*

$$\frac{mS}{P} < 2, \qquad \textit{and hence} \qquad S < \frac{2P}{m}.$$

Proof. In a single column write its entries as $b_1, \ldots, b_m$, with indices interpreted cyclically modulo m. Since every entry is at least 1 and $m \geq 3$, the two entries b_{i+1}, b_{i+2} are distinct positions and neither is b_i. Therefore

$$\frac{b_i}{\prod_{k=1}^{m} b_k} = \frac{1}{\prod_{k \neq i} b_k} \leq \frac{1}{b_{i+1} b_{i+2}} \leq \frac{1}{2}\left(\frac{1}{b_{i+1}^2} + \frac{1}{b_{i+2}^2}\right).$$

Summing over i, each reciprocal square occurs twice with coefficient $1/2$. Summing next over columns gives

$$\frac{mS}{P} \leq \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{1}{a_{ij}^2} < 2,$$

where the final step is the same finite distinct-integer bound as above. □

The original two-row verification control

The source supplies the following matrix [1]:

$$\begin{pmatrix} 2 & 4 & 7 & 24 & 40 & 70 & 105 & 140 & 168 & 280 \\ 420 & 210 & 120 & 35 & 21 & 12 & 8 & 6 & 5 & 3 \end{pmatrix}.$$

Direct arbitrary-precision computation gives row sums $(840, 840)$, ten column products all equal to 840, and 20 distinct positive entries. The matrix is included as `certificates/source_two_rows.json`.

For two rows, the relevant identity is instead

$$2 = \sum_j \frac{a_{1j} + a_{2j}}{a_{1j} a_{2j}} = \sum_j \left(\frac{1}{a_{1j}} + \frac{1}{a_{2j}}\right).$$

This is a sum of reciprocals, not reciprocal squares. It is exactly 2 for the source control, and the obstruction above does not apply.

Why the hypotheses matter

Global distinctness is essential: the 3×4 matrix whose every entry is 2 has every row sum and column product equal to 8. It is rejected because its entries repeat. Ordinary sum–product equality is weaker: the source's matrix $\left(\begin{smallmatrix} 3 & 21 \\ 14 & 10 \\ 20 & 4 \end{smallmatrix}\right)$ has common row sum 24 and common column product 840, not the same value. Neither example contradicts Theorem 1.

There is no hidden scaling step. Scaling all entries by c changes a row sum by c and an m-entry column product by c^m. For example, doubling the two-row control gives sum 1680 and product 3360.

With $n \geq 2$, the one-row case is impossible as well: the column condition forces every entry to be N, so its row sum would be $nN > N$. Thus among positive row counts, the equality case can occur only with two rows.

3 Verification, reproducibility, and provenance

The bundle uses Python's standard library only. The matrix checker recomputes dimensions, integer positivity, global distinctness, every row sum, and every column product directly from JSON entries. It does not call a search solver or use the impossibility theorem to decide validity.

The executed validation consists of:

Check	Executed scope; all passed
Checker regression	33 tests, including malformed inputs, repeated entries, corruptions, wrong N, scaling, ordinary-SP controls, and large integers.
Symbolic algebra	Exact integer-polynomial coefficient comparison; the sum-of-squares identity has zero residual.
Ordered triples	All 64,000 triples in $\{1, \ldots, 40\}^3$, including repeats.
Distinct sets	All 4,095 nonempty subsets of $\{1, \ldots, 12\}$.
Telescoping	Identity and reciprocal-square bound for $1 \leq M \leq 512$.
Higher-row arrays	360 arrays, 2,340 columns: $3 \leq m \leq 12$, $1 \leq n \leq 12$, three trials per shape. Consecutive values and seeded samples from 1 through 10,000; seed `20260914`.

An additional exact-rational probe uses 101-digit entries. A separate audit script, which does not import the matrix checker, rechecks the two-row control and the repeated-entry boundary example.

The finite tests are not the proof. Theorem 1 proves nonexistence for all permitted dimensions and values. The tests are regression and transcription safeguards. No exhaustive matrix or factorization search was needed or performed. No proof-assistant verification or external reviewer is claimed.

Reproduction

From the extracted bundle directory, run:

```
python3 src/reproduce.py
```

This runs both the target checker and the explicit two-row control override. Standalone checker commands are in `README.md`. Actual commands, return codes, versions, and inspected output are in `logs/`. The note's LaTeX source and a SHA-256 manifest are included.

Source and priority audit

The original page and the separate unsolved-problems list were inspected on 14 September 2026; the latter still displayed this statement as a conjecture [2]. The targeted searches recorded in `references/source_audit.md` did not locate a later proof or construction. This is not an exhaustive certification of historical priority. It is separate from the mathematical conclusion proved here. No contact or publication was undertaken.

References

[1] Erich Friedman, *Problem of the Month (November 2002)*, Math Magic. Attributes the equality-case conjecture and two-row example to Joseph DeVincentis. Accessed 14 September 2026.
`https://erich-friedman.github.io/mathmagic/1102.html`

[2] Erich Friedman, *Unsolved Problems from Math Magic*, item 8. Accessed 14 September 2026.
`https://erich-friedman.github.io/mathmagic/unsolved.html`