Eternal domination and the Lovász theta function: an unbounded separation and a smallest counterexample

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Abstract

We show that the Lovász theta function can exceed the standard one-guard eternal domination number by an unbounded factor. Combining results of Alon and of Goddard, Hedetniemi, and Hedetniemi gives an explicit infinite family (H_k) with

$$2 \le \gamma^{\infty}(H_k) \le 3 \quad \text{and} \quad \vartheta(H_k) = \Theta\Big(|V(H_k)|^{1/3}\Big).$$

Thus the ratio $\vartheta/\gamma^{\infty}$ is unbounded. Second, let G be the ten-vertex graph with graph6 record `IEhbtj{ro`. MacGillivray, Mynhardt, and Virgile proved that $\gamma^{\infty}(G) = 3$. We give an exact rational feasible matrix for the standard Lovász-theta semidefinite program with objective

$$\frac{7593}{2500} = 3.0372.$$

Their exhaustive result through order nine, together with the theta sandwich inequality, shows that G has minimum possible order among counterexamples. A solver-free verifier checks the finite theta certificate and independently recomputes the eternal-domination game.

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1 Introduction and conventions

In the one-guard eternal domination game on a finite simple graph G, at most one guard occupies each vertex and the occupied vertices form a dominating set. At each turn an unoccupied vertex is attacked, and exactly one guard must move along an edge to the attacked vertex so that the new guard set is again dominating. The minimum number of guards that can defend indefinitely is $\gamma^{\infty}(G)$ [3]. The elementary bounds for this game are

$$\alpha(G) \le \gamma^{\infty}(G) \le \mathrm{cc}(G), \tag{1}$$

where $\mathrm{cc}(G)$ denotes the clique-cover number [4, Theorem 1].

West's problem page asks whether γ^{∞} is bounded below by the Lovász theta function [7]. There is a potential notation trap. Work on eternal domination often writes $\theta(G)$ for the clique-cover number. Here we instead write

$$\mathrm{cc}(G) = \chi(\overline{G})$$

for clique cover and reserve $\vartheta(G)$ for the Lovász theta function. Lovász introduced the function in 1979 [5]. With the standard convention used by Alon and Kahale [2],

$$\vartheta(G) = \max\left\{\langle J, X\rangle : X \succeq 0,\ \operatorname{Tr}(X) = 1,\ X_{ij} = 0 \text{ for } ij \in E(G)\right\}. \tag{2}$$

It satisfies the familiar sandwich

$$\alpha(G) \le \vartheta(G) \le \chi(\overline{G}) = \operatorname{cc}(G). \tag{3}$$

We give two complementary forms of the negative answer. Section 2 records an unbounded polynomial separation that follows directly from established results. Section 3 gives an exact smallest counterexample, together with a solver-free certificate.

2 An explicit unbounded separation

Alon's explicit Ramsey construction and a theorem on eternal security combine to give considerably more than a single counterexample.

Theorem 1. *For every integer $k \ge 2$ not divisible by three, put $n_k = 2^{3k}$. There is an explicit graph H_k on n_k vertices such that*

$$\alpha(H_k) = 2, \qquad 2 \le \gamma^\infty(H_k) \le 3, \qquad \vartheta(H_k) = \Theta(n_k^{1/3}),$$

and hence

$$\frac{\vartheta(H_k)}{\gamma^\infty(H_k)} = \Theta(n_k^{1/3}).$$

In particular, ϑ/γ^∞ is unbounded, and there is no constant C such that $\vartheta(H) \le C\gamma^\infty(H)$ for every graph H.

Proof. Alon explicitly constructs a $2^{k-1}(2^{k-1}-1)$-regular triangle-free Cayley graph F_k on n_k vertices whose complement satisfies

$$\vartheta(\overline{F_k}) = \Theta(n_k^{1/3})$$

as k ranges over these values [1, Corollary 2.3]. Set $H_k = \overline{F_k}$. Since $k \ge 2$, the graph F_k has an edge, and triangle-freeness gives

$$\alpha(H_k) = \omega(F_k) = 2.$$

Goddard, Hedetniemi, and Hedetniemi proved that every graph with independence number two has one-guard eternal domination number at most three [4, Theorem 4]. Their notation σ_1 denotes the parameter written γ^∞ here. Equation (1) therefore gives

$$2 \le \gamma^\infty(H_k) \le 3.$$

Dividing Alon's theta estimate by a quantity bounded between two and three gives the asserted ratio. □

3 A smallest counterexample

MacGillivray, Mynhardt, and Virgile found two smallest graphs for which $\gamma^\infty < \operatorname{cc}$ [6]. Their Table 9 records the graph6 strings `IEhbtj{ro` and `IEhbtn{ro`. Their Fact 5.1 gives clique-cover number four, while Proposition 5.1 and its proof give eternal domination number three. The observation here is that the 26-edge member of that pair is already a counterexample to the proposed Lovász-theta lower bound.

Let G be the graph on vertices $0, \dots, 9$ whose graph6 record is

`IEhbtj{ro`.

For complete independence from graph6 software, its edge set is

$$E(G) = \{03, 04, 07, 08, 09,\ 13, 15, 16, 18, 19,\ 24, 25, 26, 27, 28,\\ 36, 37, 38,\ 46, 48, 49,\ 57, 58, 59,\ 69, 79\},$$

where ij denotes the unordered edge $\{i, j\}$.

Theorem 2. *The graph G satisfies*

$$\gamma^{\infty}(G) = 3 < \frac{7593}{2500} \le \vartheta(G).$$

Consequently, the inequality $\gamma^{\infty}(H) \ge \vartheta(H)$ does not hold for all graphs H.

Proof. The equality $\gamma^{\infty}(G) = 3$ is established in Proposition 5.1 and its proof by MacGillivray, Mynhardt, and Virgile [6]. We also independently verify it by exact state enumeration in Section 4.

For the theta bound, define $X = M/10000$, where

$$M = \begin{pmatrix} 1170 & 576 & 621 & 0 & 0 & 576 & 621 & 0 & 0 & 0 \\ 576 & 1172 & 621 & 0 & 576 & 0 & 0 & 621 & 0 & 0 \\ 621 & 621 & 1025 & 335 & 0 & 0 & 0 & 0 & 0 & 512 \\ 0 & 0 & 335 & 1025 & 621 & 621 & 0 & 0 & 0 & 512 \\ 0 & 576 & 0 & 621 & 1172 & 576 & 0 & 621 & 0 & 0 \\ 576 & 0 & 0 & 621 & 576 & 1172 & 621 & 0 & 0 & 0 \\ 621 & 0 & 0 & 0 & 0 & 621 & 1025 & 335 & 512 & 0 \\ 0 & 621 & 0 & 0 & 621 & 0 & 335 & 1025 & 512 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 512 & 512 & 607 & 196 \\ 0 & 0 & 512 & 512 & 0 & 0 & 0 & 0 & 196 & 607 \end{pmatrix}.$$

The matrix is symmetric, its diagonal entries sum to 10000, and direct inspection against the displayed edge list gives $M_{ij} = 0$ for every $ij \in E(G)$.

It remains to verify positive semidefiniteness without numerical approximation. Exact Gaussian elimination gives the following diagonal pivots in an LDL^{T} factorization of M:

$$1170,\quad \frac{57748}{65},\quad \frac{33696545}{57748},\quad \frac{5611637865}{6739309},\quad \frac{5875307251}{81328085},\\ \frac{147954747608}{2670594205},\quad \frac{774768046339541}{1627502223688},\quad \frac{47387055721842240}{774768046339541},\\ \frac{1491051941429}{39596916715},\quad \frac{50290089836283}{1491051941429}.$$

Every pivot is positive, so M, and hence X, is positive definite. Thus X is feasible in (2). Finally, the sum of all entries of M is 30372, and therefore

$$\langle J, X \rangle = \frac{30372}{10000} = \frac{7593}{2500} > 3.$$

Feasibility gives the asserted lower bound for $\vartheta(G)$, and the strict separation follows. □

Corollary 3. *The minimum order of a graph H satisfying $\gamma^{\infty}(H) < \vartheta(H)$ is ten.*

Proof. MacGillivray, Mynhardt, and Virgile computationally verified that every graph of order at most nine satisfies $\gamma^{\infty}(H) = \mathrm{cc}(H)$ [6, Proposition 5.2]. By (3), every such graph satisfies

$$\vartheta(H) \le \mathrm{cc}(H) = \gamma^{\infty}(H).$$

Theorem 2 supplies a counterexample of order ten. □

4 Independent exact verification

The accompanying verifier uses Python's standard library and no SDP solver. It parses the graph6 record independently in two implementations and compares the result with the frozen edge list above. For the theta certificate it checks symmetry, trace, all 26 edge zeros, the exact entry sum, and the ten rational LDL^{T} pivots.

For the eternal-domination game, let $\mathcal{D}_k$ be the dominating k-subsets of $V(G)$. Starting with $\mathcal{F}_0 = \mathcal{D}_k$, repeatedly set

$$\mathcal{F}_{t+1} = \left\{ S \in \mathcal{F}_t : \begin{array}{l} \text{for every } v \notin S \text{ there is } u \in S \cap N(v) \\ \text{such that } (S \setminus \{u\}) \cup \{v\} \in \mathcal{F}_t \end{array} \right\}.$$

Because the state space is finite, this deletion process reaches the greatest closed family. A nonempty closed family is equivalent to an eternal defense: it supplies a response after every attack, while the reachable states of any successful defense form such a family.

The exact fixed-point sizes for $k = 1, 2, 3$ are

$$0, \quad 0, \quad 86.$$

There are 86 dominating triples, all of which survive the fixed point. The saved certificate lists them and the verifier checks a legal successor inside the family for all $86(10-3) = 602$ configuration-attack pairs. This independently reproduces $\gamma^{\infty}(G) = 3$.

5 Attribution, novelty, and scope

The ingredients of Theorem 1 are established results: Alon supplies the explicit theta construction, and Goddard, Hedetniemi, and Hedetniemi supply the constant eternal-domination bound. The theorem is a direct synthesis, not a new graph construction. Likewise, the ten-vertex graph and its eternal-domination value are due to MacGillivray, Mynhardt, and Virgile [6]. The candidate contribution is recognizing that these results answer West's Lovász-theta question, together with the exact rational certificate and minimum-order identification.

A focused search through notation variants, the cited papers' reference and citation graphs, recent related papers, public problem pages, arXiv, and scholarly indexes found no prior public source explicitly making either eternal-domination connection as of 26 July 2026. That search cannot establish worldwide priority: an unindexed, unpublished, or differently phrased antecedent may exist.

The counterexample theorem is an exact finite result. The minimum-order corollary additionally relies on the published exhaustive computation through order nine. The unbounded-family theorem relies on the cited published results. No numerical estimate of the optimum theta value of the ten-vertex graph is needed.

AI-assistance and authorship disclosure

The observation, exact certificates, independent verifiers, literature review, proof drafting, and publication materials were developed with heavy assistance from ChatGPT 5.6 Sol under Alec Kriebel's direction. Alec Kriebel is the named human author. No external expert was contacted or has reviewed this note. Passing the supplied checks is not peer review.

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