Low-Schmidt rigidity and tensor-local constraints in the exceptional unitary Hecke Yang–Baxter class

Alec Kriebel*

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Abstract

Let $q = e^{i\pi/3}$. The unresolved exceptional family of unitary two-eigenvalue Yang–Baxter matrices has parameters

$$\left[q, \frac{1}{2}, d\right],$$

with existence known when $4 \mid d$ and open when $d \equiv 2 \pmod 4$. We derive exact structural constraints on this unresolved spectrum.

First, every matrix in the exceptional class is automatically standard and faithfully represents the Jones–Wenzl trace quotient $H_n(3, 6)$. Its simple tensor-space multiplicities are

$$m_{\lambda,n} = D_\lambda \left(\frac{d}{2}\right)^n, \qquad D_\lambda \in \{1, 2, 3\}.$$

Thus all simple-multiplicity, central-rank, Markov-weight, and branching arithmetic permits every even d. Any further divisibility obstruction must use the repeated realization of adjacent generators by one common two-site tensor.

We then prove low-Schmidt rigidity: an exceptional reflection of operator-Schmidt rank at most three exists exactly when $4 \mid d$. At rank four, projecting the full cubic relation modulo the intrinsic Schmidt supports yields two joint-sandwich identities. If either associated sandwich map is injective, then $4 \mid d$; hence every rank-four candidate in $d \equiv 2 \pmod 4$ has nonzero Hermitian traceless common annihilators on both tensor legs.

Finally, any hypothetical dimension-six solution is nonrestrictable, satisfies

$$\mathcal{C}_L(P) \cap \mathcal{C}_R(P) = \mathbb{C}\mathbf{1}_6,$$

and lies outside the primitive-Weyl Bell-diagonal and four-product Clifford-frame mechanisms. A four-dimensional one-sided square restriction is not excluded. Thus the complete spectrum remains open. Any dimension-six witness must have $\mathrm{OSR}(H) \geq 4$, and at rank four it must exhibit simultaneous sandwich degeneracy.

Status. This is an unrefereed, AI-assisted research note. The proofs, exact verifiers, adversarial audits, and computational provenance are public. Priority claims are provisional and no specialist peer review is claimed.

1 Introduction and main results

Let $V \cong \mathbb{C}^d$. A unitary $R \in \mathrm{End}(V \otimes V)$ is a Yang–Baxter operator when

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23} \quad \text{on } V^{\otimes 3}. \tag{1}$$

*This work was developed with substantial assistance from OpenAI ChatGPT. The exact scope of that assistance is disclosed at the end of the paper.

Lechner [6] classified the possible characters of unitary Yang–Baxter matrices with two eigenvalues, leaving one potentially empty family:

$$\left[e^{i\pi/3}, \frac{1}{2}, d\right], \qquad d > 2 \text{ even}.$$

The notation records the nonnegative Hecke phase, the normalized rank of the (-1)-spectral projection, and the local dimension.

The dimension-two class is empty. In [5] we constructed an exact 16×16 unitary representative in local dimension four. Identity stabilization then gives representatives in every dimension divisible by four. The unresolved part of the spectrum is therefore

$$d = 6, 10, 14, \ldots.$$

This question is also the strict ordinary-localization problem associated with the $\mathcal{C}(\mathfrak{sl}_3, 6)$ Jones–Wenzl braid representations studied by Galindo, Hong, and Rowell [4].

It is tempting to seek the missing factor of two in the representation theory of $H_n(3,6)$. One of the main conclusions of this paper is that this cannot work at the level of abstract simple multiplicities, central ranks, or their branching recurrences: all of that arithmetic is compatible with every even local dimension. The extra restriction, if it exists, must come from realizing every adjacent Hecke generator by the two shifts of one common two-site matrix.

We summarize the results in two theorems. Precise definitions appear in Section 2.

Theorem 1.1 (Automatic tower structure and its limitation)**.** *Let $P \in \mathrm{End}(V \otimes V)$ be an exceptional projection in local dimension d. Then:*

(i)

$$\mathrm{Tr}_1 P = \mathrm{Tr}_2 P = \frac{d}{2}\mathbf{1}_d.$$

No finite-dimensional irreducibility assumption is needed.

(ii) *The normalized tensor trace is the $\eta = 1/2$ Markov trace at every level, and the local Hecke representation induces an injection*

$$H_n(3,6) \longrightarrow \mathrm{End}(V^{\otimes n}).$$

Equivalently, the kernel of the raw Hecke representation is exactly the Markov-trace annihilator.

(iii) *If $S_{\lambda,n}$ is a simple $H_n(3,6)$-module and $D_\lambda \in \{1,2,3\}$ is the associated quantum dimension, then its tensor-space multiplicity is*

$$m_{\lambda,n} = D_\lambda \left(\frac{d}{2}\right)^n.$$

These multiplicities and all their restriction recurrences are integral exactly when d is even.

Thus ordinary Hecke simple-block, central-rank, Markov-trace, and branching arithmetic cannot force $4 \mid d$.

Theorem 1.2 (Tensor-local frontier)**.** *Let $H = \mathbf{1} - 2P$ be the reflection associated with an exceptional projection.*

(i) *If a rank-r projection belongs to either one-leg commutant of P, then*

$$8 \mid rd^2.$$

(ii) *An exceptional reflection with* $\mathrm{OSR}(H) \le 3$ *exists in dimension d if and only if $4 \mid d$. Hence every hypothetical solution in $d \equiv 2 \pmod 4$ has* $\mathrm{OSR}(H) \ge 4$.

(iii) *Suppose* $\mathrm{OSR}(H) = 4$*, with intrinsic Schmidt supports $\mathcal{A}, \mathcal{B}$. If either joint-sandwich map*

$$\mathfrak{S}_{\mathcal{B}|\mathbb{C}\mathbf{1}+\mathcal{A}} \quad or \quad \mathfrak{S}_{\mathcal{A}|\mathbb{C}\mathbf{1}+\mathcal{B}}$$

is injective, then $4 \mid d$. Consequently, in $d \equiv 2 \pmod 4$, both maps have nonzero Hermitian traceless coefficient-matrix annihilators.

(iv) *Every square-invariant local subspace inherits a balanced exceptional solution and has even dimension. A least-dimensional solution with $d \equiv 2 \pmod 4$ is nonrestrictable. In particular, no dimension-six solution is restrictable.*

(v) *In dimension six,*

$$\mathcal{C}_L(P) \cap \mathcal{C}_R(P) = \mathbb{C}\mathbf{1}_6.$$

This does not assert that either individual leg commutant is scalar.

(vi) *A primitive-Weyl Bell-diagonal exceptional reflection, and every reflection in the four-product Clifford-frame class defined in Section 8, has $4 \mid d$.*

Theorem 1.2(iii) is the outcome of a final bounded attack on unrestricted operator-Schmidt rank four. It uses the full cubic, not a Clifford or symmetry ansatz. It does not prove that either sandwich map is always injective.

The exact dimension-six conclusion is therefore narrower than "tensor-locally irreducible." A rank-two square-invariant subspace is impossible, while a rank-four one-sided square-invariant subspace remains possible in principle; if it exists, its two-dimensional orthogonal complement cannot also have invariant tensor square.

2 Exceptional projections and conventions

Fix throughout

$$q = e^{i\pi/3}.$$

An *exceptional projection* is an orthogonal projection

$$P = P^* = P^2 \in \mathrm{End}(V \otimes V), \qquad \operatorname{rank} P = \frac{d^2}{2},$$

such that, with $P_1 = P \otimes \mathbf{1}$ and $P_2 = \mathbf{1} \otimes P$,

$$P_1P_2P_1 - P_2P_1P_2 = \frac{1}{3}(P_1 - P_2). \tag{2}$$

The associated reflection and unitary Hecke operator are

$$H = \mathbf{1} - 2P, \qquad R = q\mathbf{1} - (1+q)P = \frac{q-1}{2}\mathbf{1} + \frac{q+1}{2}H. \tag{3}$$

A direct expansion shows that (2) is equivalent to

$$H_1H_2H_1 - H_2H_1H_2 = \frac{1}{3}(H_1 - H_2), \tag{4}$$

and that (4) is equivalent to the Yang–Baxter equation for R. Moreover,

$$H = H^*, \quad H^2 = \mathbf{1}, \quad \operatorname{Tr} H = 0,$$

and R is unitary with eigenvalues $-1, q$, each of multiplicity $d^2/2$.

The operator-Schmidt rank of $X \in M_d \otimes M_d$, denoted $\mathrm{OSR}(X)$, is the least r for which

$$X = \sum_{j=1}^{r} A_j \otimes B_j.$$

Our rank statements concern the reflection H, not R. Once automatic standardness is proved, the Schmidt supports of H consist of traceless matrices. Hence the identity term in (3) adds one independent Schmidt direction:

$$\mathrm{OSR}(R) = \mathrm{OSR}(H) + 1. \tag{5}$$

Define the two one-leg commutants

$$\begin{aligned} \mathcal{C}_L(P) &= \{x \in M_d : [x \otimes \mathbf{1}, P] = 0\}, \\ \mathcal{C}_R(P) &= \{x \in M_d : [\mathbf{1} \otimes x, P] = 0\}. \end{aligned} \tag{6}$$

These are two subalgebras of the same M_d, after identifying the two copies of the local space. Their intersection is meaningful, but it must not be confused with either algebra individually.

Definition 2.1. A nonzero subspace $W \subseteq V$ is *square-invariant* when

$$R(W \otimes W) \subseteq W \otimes W.$$

The solution is *restrictable* when there is a proper nonzero $W \subset V$ for which both $W^{\otimes 2}$ and $(W^{\perp})^{\otimes 2}$ are invariant.

Because R is unitary, every invariant tensor square is automatically reducing. Definition 2.1 agrees with the use in Conti–Lechner [3, §6.1].

3 Automatic standardness and the faithful trace quotient

The first theorem removes a potentially serious hypothesis from the dimension problem.

Theorem 3.1 (Automatic standardness)**.** *Every exceptional projection satisfies*

$$\operatorname{Tr}_1 P = \operatorname{Tr}_2 P = \frac{d}{2}\mathbf{1}_d. \tag{7}$$

Proof. The Hecke operator R in (3) has spectrum $\{-1, q\}$. It has no pair of opposite eigenvalues, since that would require $q = 1$. Lechner's no-opposite-spectrum theorem [6, Proposition 2.4 and Lemma 3.1] therefore makes the associated braid character a Markov trace. This derives the needed subfactor irreducibility; it is not an assumption on the finite matrix representation.

Lechner's partial-trace criterion [6, Proposition 2.3] says that the Markov property is equivalent to

$$\varphi(R) = \tau(R)\mathbf{1}_d, \qquad \varphi = \frac{1}{d}(\mathrm{Tr}\otimes \mathrm{id}),$$

where τ is normalized matrix trace. Put

$$\eta = \tau(P) = \frac{\operatorname{rank} P}{d^2} = \frac{1}{2}.$$

Using $R = q\mathbf{1} - (1+q)P$ and $1+q \neq 0$, comparison gives

$$\varphi(P) = \eta\mathbf{1}_d = \frac{1}{2}\mathbf{1}_d.$$

This is the first identity in (7).

For completeness, apply the same argument to the tensor-flipped operator FRF. Reversal of the three tensor positions preserves the Yang–Baxter equation, and FRF has the same spectrum. Its first partial trace is the second partial trace of R, proving the second identity. One may instead use the equality of the two partial traces proved in [3, Theorem 5.10]. □

Corollary 3.2. *Every exceptional reflection satisfies*

$$\mathrm{Tr}_1 H = \mathrm{Tr}_2 H = 0.$$

In every minimal Schmidt decomposition of H, both intrinsic Schmidt supports consist of traceless matrices.

Proof. The first claim follows from $H = \mathbf{1} - 2P$. For the second, write $H = \sum_j A_j \otimes B_j$ with each family linearly independent. Then $\mathrm{Tr}_2 H = \sum_j (\mathrm{Tr}\, B_j)A_j = 0$, so every $\mathrm{Tr}\, B_j = 0$; the other leg is identical. □

Let $H_n(q)$ be the specialized Hecke algebra generated by projections $e_1, \ldots, e_{n-1}$, with braid generators

$$g_i = q\mathbf{1} - (1+q)e_i.$$

The local representation is

$$\rho_n : H_n(q) \longrightarrow \mathrm{End}(V^{\otimes n}), \qquad e_i \longmapsto P_i.$$

Write $\tau_n = d^{-n}\,\mathrm{Tr}$ for normalized matrix trace.

Proposition 3.3 (Faithful trace quotient)**.** *For every n, $\tau_n \circ \rho_n$ is the Markov trace $\mu_{1/2}$. If*

$$\mathrm{Ann}_n = \{x \in H_n(q) : \mu_{1/2}(yx) = 0 \textit{ for every } y \in H_n(q)\},$$

then

$$\ker \rho_n = \mathrm{Ann}_n\,.$$

Consequently ρ_n induces an injection

$$H_n(3,6) = H_n(q)/\,\mathrm{Ann}_n \hookrightarrow \mathrm{End}(V^{\otimes n}).$$

Proof. For $x \in H_n(q)$, Theorem 3.1 gives

$$\begin{aligned}\tau_{n+1}((\rho_n(x) \otimes \mathbf{1})P_n) &= d^{-(n+1)}\,\mathrm{Tr}(\rho_n(x)\,\mathrm{Tr}_{n+1}(P_n)) \\ &= \frac{1}{2}\,\tau_n(\rho_n(x)).\end{aligned}$$

Thus the Markov property holds at every level; it is not inferred merely from a two-strand trace coincidence. Uniqueness identifies this trace with $\mu_{1/2}$.

Wenzl's positive-trace parameter in the present convention is

$$\eta_{\ell,k} = \frac{\sin(\pi(k-1)/\ell)}{2\cos(\pi/\ell)\sin(\pi k/\ell)}.$$

At $(\ell, k) = (6, 3)$, it is $1/2$, identifying the quotient with $H_n(3, 6)$; see [6, Theorem 3.3] and [7].

If $x \in \mathrm{Ann}_n$, choose $y = x^*$. Then

$$0 = \mu_{1/2}(x^*x) = \tau_n(\rho_n(x)^*\rho_n(x)).$$

Faithfulness of ordinary matrix trace gives $\rho_n(x) = 0$. The reverse inclusion is immediate. This proves equality of the kernel and annihilator. Notice that the raw specialized Hecke representation need not be faithful; faithfulness is asserted only after passage to the trace quotient. □

4 Complete tower multiplicities and their exact limitation

The simple $H_n(3, 6)$-modules are indexed by admissible Young diagrams. Equivalently, after subtracting the third row, their labels lie in the ten-point alcove

$$(a, b), \qquad a, b \geq 0, \quad a + b \leq 3.$$

The associated quantum dimensions are

$$D_{a,b} = \frac{\sin((a+1)\pi/6)\sin((b+1)\pi/6)\sin((a+b+2)\pi/6)}{\sin(\pi/6)^2\sin(2\pi/6)}. \tag{8}$$

Their values are

$D_{a,b}$	(a, b)
1	$(0,0), (3,0), (0,3)$
3	$(1,1)$
2	$(1,0), (0,1), (2,0), (0,2), (2,1), (1,2).$

Adding a box gives the usual alcove branching, and direct substitution in (8) yields

$$\sum_{\nu:\lambda\nearrow\nu} D_\nu = 2D_\lambda. \tag{9}$$

Theorem 4.1 (All simple multiplicities)**.** *Decompose the tensor representation as*

$$V^{\otimes n} \cong \bigoplus_\lambda S_{\lambda,n} \otimes \mathbb{C}^{m_{\lambda,n}}.$$

Then

$$\boxed{m_{\lambda,n} = D_\lambda \left(\frac{d}{2}\right)^n.} \tag{10}$$

Proof. Let $f_{\lambda,n} = \dim S_{\lambda,n}$. In the $\mathcal{C}(\mathfrak{sl}_3, 6)$ normalization, the generating object X has quantum dimension 2. The categorical trace of a minimal projection onto the simple label λ is its quantum dimension divided by 2^n; see [4, §5.6] and [7, §§2–3]. Thus the Wenzl trace weight is

$$t_{\lambda,n} = \frac{D_\lambda}{2^n};$$

the identification is $H_n(3,6) \cong \mathrm{End}_{\mathcal{C}}(X^{\otimes n})$. The recursion for these weights is exactly (9) divided by 2^{n+1}.

Under ρ_n, a minimal projection in the λ-block has ordinary matrix rank $m_{\lambda,n}$, hence normalized trace $m_{\lambda,n}/d^n$. Proposition 3.3 equates this with $D_\lambda/2^n$, proving (10). □

Corollary 4.2 (Exact arithmetic limitation)**.** *All simple multiplicities, central-idempotent ranks, and represented restriction recurrences are integral if and only if d is even. In particular, this complete simple-multiplicity, central-rank, Markov-weight, and branching arithmetic cannot distinguish $d = 6$ from $d = 4$.*

Proof. If $d = 2s$, formula (10) is $m_{\lambda,n} = D_\lambda s^n$, an integer at every level. Moreover, (9) gives

$$\sum_{\nu:\lambda\nearrow\nu} m_{\nu,n+1} = d\, m_{\lambda,n},$$

which is exactly the tensor-space restriction recurrence. If d is odd, already the two-strand negative spectral rank $d^2/2$ is not an integer. □

Remark 4.3 (What the limitation proves). For $d = 2s$, the abstract modules

$$\bigoplus_\lambda S_{\lambda,n} \otimes \mathbb{C}^{D_\lambda s^n}$$

have the required dimensions, traces, and restriction multiplicities. At $d = 6$, the three-strand common-one, common-zero, and generic two-projection multiplicities are $27, 27, 81$. Exact four-strand models also realize the expected simple corners and pairings with the odd multiplicity 81.

These are abstract Hecke modules, not a Yang–Baxter matrix on $\mathbb{C}^6 \otimes \mathbb{C}^6$. They do not realize $P_{12} = P \otimes \mathbf{1}_6$ and $P_{23} = \mathbf{1}_6 \otimes P$ for one common two-site projection. The insufficiency conclusion is therefore precisely that ordinary Hecke block data and their branching cannot prove four-divisibility; strict same-P tensor coherence remains available as an obstruction.

5 Leg-commutant arithmetic and low-Schmidt rigidity

We first isolate the arithmetic consequence of a genuine one-leg projection. This is the bridge from tensor-local structure to four-divisibility.

Lemma 5.1 (Two exceptional projections)**.** *Let a, b be projections on a D-dimensional Hilbert space such that*

$$aba - bab = \frac{1}{3}(a - b),$$

$$\operatorname{rank} a = \operatorname{rank} b = \frac{D}{2}, \qquad \operatorname{Tr}(ab) = \frac{D}{4}.$$

Then

$$\dim(\operatorname{ran} a \cap \operatorname{ran} b) = \dim(\ker a \cap \ker b) = \frac{D}{8}.$$

Proof. Put
$$z = aba - \frac{1}{3}a = bab - \frac{1}{3}b.$$
Projection identities give $az = za = z$ and $bz = zb = z$. Thus the range of z lies in the common range of a, b, where $z = 2\mathbf{1}/3$. Self-adjointness therefore gives $z = (2/3)e$, with e the common-range projection. Taking traces,
$$\operatorname{Tr} e = \frac{\operatorname{Tr}(ab) - \frac{1}{3}\operatorname{Tr} a}{2/3} = \frac{D}{8}.$$
Equal half ranks make the common-range and common-kernel dimensions equal; this also follows by applying the same argument to $\mathbf{1} - a, \mathbf{1} - b$. □

Theorem 5.2 (Invariant leg projection)**.** *If a rank-r projection z belongs to either $\mathcal{C}_L(P)$ or $\mathcal{C}_R(P)$, then*
$$\boxed{8 \mid rd^2.}$$

Proof. Suppose $z \in \mathcal{C}_L(P)$, put $W = zV$, and let Q be the restriction of P to $W \otimes V$. Theorem 3.1 gives
$$\operatorname{rank} Q = \operatorname{Tr}((z \otimes \mathbf{1})P) = \operatorname{Tr}(z\ \operatorname{Tr}_2 P) = \frac{rd}{2}.$$
Restrict P_1, P_2 to $W \otimes V \otimes V$. The resulting projections $a = Q \otimes \mathbf{1}$, $b = \mathbf{1}_W \otimes P$ act on a space of dimension $D = rd^2$, obey the exceptional relation, and both have rank $D/2$. Their overlap trace is
$$\begin{aligned}\operatorname{Tr}(ab) &= \operatorname{Tr}_{W\otimes V}[Q(\mathbf{1}_W \otimes \operatorname{Tr}_2 P)] \\ &= \frac{d}{2}\operatorname{Tr} Q = \frac{rd^2}{4}.\end{aligned}$$
Lemma 5.1 makes their common-range dimension $rd^2/8$, an integer. The right-leg case follows after tensor flip. □

Corollary 5.3. *If $d \equiv 2 \pmod 4$, every projection in either leg commutant has even rank. In particular, either leg containing a rank-one projection forces $4 \mid d$.*

We next show that low operator-Schmidt rank supplies such a projection. The subtle point is that the known rank-two and rank-three unitary structure theorems use four independent local pre- and post-unitaries. That equivalence does not preserve the Yang–Baxter equation.

Theorem 5.4 (Twisted control)**.** *Let $H \in \operatorname{End}(V \otimes V)$ be a Hermitian unitary which is locally equivalent, through independent product unitaries before and after, to a controlled bipartite unitary. Suppose*
$$H_1H_2H_1 - H_2H_1H_2 = c(H_1 - H_2) \tag{11}$$
with $|c| \neq 1$. Then one true one-leg commutant of H contains a rank-one projection.

Proof. Assume first that the supplied control is on the first factor. There are unitaries Q, R, S, T, rank-one coordinate projections E_i, and unitaries U_i such that
$$H = (Q \otimes S)\left(\sum_i E_i \otimes U_i\right)(R \otimes T). \tag{12}$$

We do not insert the controlled operator in parentheses into (11). Instead, apply the valid same-site conjugacy

$$\widetilde{H} = (Q^* \otimes Q^*)H(Q \otimes Q).$$

It preserves (11) and gives the twisted controlled form

$$\widetilde{H} = \sum_i E_i K \otimes V_i, \qquad K = RQ, \quad V_i = Q^* S U_i T Q. \tag{13}$$

The ij block of $\widetilde{H}$ is $K_{ij}V_i$. Hermiticity is therefore

$$K_{ij}V_i = \overline{K_{ji}}V_j^*. \tag{14}$$

Define the support graph of K to have vertices i and an edge i–j whenever $K_{ij} \neq 0$. Equation (14) makes it undirected. Loops are retained and count as odd cycles. Its component projections reduce K, hence reduce $\widetilde{H}$ on the first leg. Along each edge the projective classes of the target unitaries alternate between $[W]$ and $[W^*]$.

If a connected component of this graph is nonbipartite, an odd cycle identifies these two classes. On that component,

$$\widetilde{H}_C = A_C \otimes W$$

for a unitary A_C. A rank-one spectral projection of the normal matrix A_C, extended by zero across the other graph components, commutes with the full $\widetilde{H}$ on its first leg.

Suppose instead that every component $C = L_C \sqcup R_C$ is bipartite. On this component,

$$K_C = \begin{pmatrix} 0 & K_{L_C R_C} \\ K_{R_C L_C} & 0 \end{pmatrix}.$$

Unitarity makes the off-diagonal blocks square unitaries, so the two parts have equal dimension. Equation (14) says that, after absorbing phases, all V_i on one part are multiples of a unitary W_C and all those on the other part are multiples of W_C^*. Consequently

$$\widetilde{H}_C = J_C \otimes W_C + J_C^* \otimes W_C^*,$$

where $J_C : R_C \to L_C$ is unitary and vanishes on the orthogonal complement. Choose an orthonormal basis $r_{C,u}$ of R_C and put $\ell_{C,u} = J_C r_{C,u}$. Pairing $r_{C,u}$ with $\ell_{C,u}$ over all components puts $\widetilde{H}$ in the fixed-point-free form

$$\widetilde{H} = \sum_x |\bar{x}\rangle\langle x| \otimes U_x, \qquad U_{\bar{x}} = U_x^*, \tag{15}$$

where $x \mapsto \bar{x}$ is an involution without fixed points.

Apply the first-leg functional $\varphi_x(A) = \langle \bar{x}|A|x\rangle$ to (11). Products of two first-leg coefficients in (15) are diagonal whenever nonzero. Hence the $\widetilde{H}_1\widetilde{H}_2\widetilde{H}_1$ coefficient vanishes, whereas the other cubic word contributes

$$\widetilde{H}(U_x \otimes \mathbf{1})\widetilde{H}.$$

The two linear terms contribute $U_x \otimes \mathbf{1}$ and zero. We obtain

$$-\widetilde{H}(U_x \otimes \mathbf{1})\widetilde{H} = c(U_x \otimes \mathbf{1}),$$

which equates a unitary with an operator of norm $|c| \neq 1$, a contradiction. Thus a nonbipartite component exists and yields the claimed rank-one projection. Conjugating back preserves its rank.

If the original controlled side is the second factor, tensor flip interchanges the two legs and preserves (11); the same argument applies. □

Cohen–Yu prove that every bipartite unitary of operator-Schmidt rank two is locally equivalent to a controlled unitary [2, Theorem 6]. Chen–Yu prove the analogous statement in rank three [1, Theorem 11]. Their hypotheses allow arbitrary finite bipartite dimensions and the four-local equivalence used in (12).

Theorem 5.5 (Complete low-Schmidt spectrum)**.** *There exists an exceptional reflection with* $\mathrm{OSR}(H) \le 3$ *in local dimension d if and only if*

$$\boxed{4 \mid d.}$$

Proof. For necessity, apply Theorem 5.4 with $c = 1/3$ and then Corollary 5.3. Rank one is immediate, and ranks two and three are covered by the cited controlled-unitary theorems.

For sufficiency, the published $d = 4$ reflection has operator-Schmidt rank three [5]. If $d = 4m$, identity stabilization replaces each Schmidt factor A_j, B_j by $A_j \otimes \mathbf{1}_m, B_j \otimes \mathbf{1}_m$, preserving both the exceptional relations and Schmidt rank three. □

Remark 5.6. Theorem 5.5 does not claim that four-local equivalence preserves Yang–Baxter locality. The only such transformation used in the proof is the same-site conjugacy leading to (13). Hermiticity and the cubic relation then provide the new argument.

6 The unrestricted Schmidt-rank-four sandwich obstruction

Let H be an exceptional reflection with $\mathrm{OSR}(H) = 4$. Its intrinsic left and right Schmidt supports are the slice spaces

$$\mathcal{A} = \{(\mathrm{id} \otimes \omega)(H) : \omega \in M_d^*\},$$
$$\mathcal{B} = \{(\omega \otimes \mathrm{id})(H) : \omega \in M_d^*\}.$$

They are four-dimensional and independent of a chosen Schmidt decomposition. Put

$$\mathcal{O}_{\mathcal{A}} = \mathbb{C}\mathbf{1} + \mathcal{A}, \qquad \mathcal{O}_{\mathcal{B}} = \mathbb{C}\mathbf{1} + \mathcal{B}.$$

For $T \in M_d$, write $[T]_{\mathcal{A}}$ and $[T]_{\mathcal{B}}$ for its cosets in $M_d/\mathcal{O}_{\mathcal{A}}$ and $M_d/\mathcal{O}_{\mathcal{B}}$, respectively.

For subspaces $\mathcal{X}, \mathcal{Y} \subseteq M_d$, define the intrinsic *joint-sandwich map*

$$\mathfrak{S}_{\mathcal{Y}|\mathcal{X}} : \mathcal{Y} \otimes \mathcal{Y} \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathcal{X}, M_d), \qquad \sum_{\ell} y_\ell \otimes z_\ell \longmapsto \left[x \mapsto \sum_{\ell} y_\ell x z_\ell \right]. \tag{16}$$

Because H is Hermitian, a real singular-value decomposition in the real Hilbert spaces of Hermitian matrices gives

$$H = \sum_{i=1}^{4} \sigma_i A_i \otimes B_i, \qquad \sigma_i > 0, \tag{17}$$

where A_i, B_i are Hermitian and Hilbert–Schmidt orthonormal. Corollary 3.2 makes every A_i, B_i traceless. Thus $\mathcal{O}_{\mathcal{A}}, \mathcal{O}_{\mathcal{B}}$ are five-dimensional operator systems.

Lemma 6.1 (Full-cubic quotient identities)**.** *For every $x \in \mathcal{O}_{\mathcal{A}}$ and $y \in \mathcal{O}_{\mathcal{B}}$,*

$$\sum_{i,k} \sigma_i \sigma_k [A_i A_k]_{\mathcal{A}} \otimes B_i x B_k = 0 \quad \textit{in } (M_d/\mathcal{O}_{\mathcal{A}}) \otimes M_d, \tag{18}$$

$$\sum_{i,k} \sigma_i \sigma_k A_i y A_k \otimes [B_i B_k]_{\mathcal{B}} = 0 \quad \textit{in } M_d \otimes (M_d/\mathcal{O}_{\mathcal{B}}). \tag{19}$$

Proof. It is clearest to begin with an arbitrary minimal factorization

$$H = \sum_{i=1}^{4} a_i \otimes b_i.$$

The two cubic words expand as

$$H_{12}H_{23}H_{12} = \sum_{i,j,k} a_i a_k \otimes b_i a_j b_k \otimes b_j,$$
$$H_{23}H_{12}H_{23} = \sum_{i,j,k} a_j \otimes a_i b_j a_k \otimes b_i b_k.$$

Project the first tensor leg of (4) to $M_d/(\mathbb{C}\mathbf{1} + \mathcal{A})$. The second cubic word and both linear terms vanish there. Applying an arbitrary functional to the third leg, and using that the b_j and a_j are bases of the two Schmidt supports, gives

$$\sum_{i,k} [a_i a_k]_{\mathcal{A}} \otimes b_i x b_k = 0 \qquad (x \in \mathcal{A}).$$

Projecting the first leg of $H^2 = \mathbf{1}$ supplies the same identity at $x = \mathbf{1}$. This proves the all-x statement on $\mathcal{O}_{\mathcal{A}}$. Tensor reversal gives the symmetric identity. Substituting (17) yields (18)–(19). □

Theorem 6.2 (Joint-sandwich divisibility)**.** *If either*

$$\mathfrak{S}_{\mathcal{B}|\mathcal{O}_{\mathcal{A}}} \quad \textit{or} \quad \mathfrak{S}_{\mathcal{A}|\mathcal{O}_{\mathcal{B}}}$$

is injective, then $4 \mid d$.

Proof. Suppose $\mathfrak{S}_{\mathcal{B}|\mathcal{O}_{\mathcal{A}}}$ is injective. Apply a linear functional φ on $M_d/\mathcal{O}_{\mathcal{A}}$ to the first factor of (18). The coefficient matrix

$$c_{ik} = \sigma_i \sigma_k \varphi([A_i A_k]_{\mathcal{A}})$$

satisfies

$$\sum_{i,k} c_{ik} B_i x B_k = 0 \qquad (x \in \mathcal{O}_{\mathcal{A}}).$$

Injectivity makes every c_{ik} zero. Since φ was arbitrary,

$$A_i A_k \in \mathbb{C}\mathbf{1} + \mathcal{A} \quad \text{for all } i, k.$$

Therefore the five-dimensional operator system $\mathbb{C}\mathbf{1} + \mathcal{A}$ is closed under multiplication. It is a finite-dimensional C^*-subalgebra of M_d.

The only complex C^*-algebras of dimension five are

$$\mathbb{C}^5, \qquad M_2(\mathbb{C}) \oplus \mathbb{C},$$

because their dimensions are sums of squares. Every faithful unital concrete representation of either algebra has a rank-one projection in its commutant: take a rank-one subprojection in a nonzero multiplicity space, using the nonzero scalar summand in the second case.

Independence of the B_i gives

$$[X \otimes \mathbf{1}, H] = 0 \quad \Longleftrightarrow \quad [X, A_i] = 0 \text{ for every } i.$$

Thus this rank-one projection lies in $\mathcal{C}_L(P)$. Theorem 5.2 forces $8 \mid d^2$, hence $4 \mid d$. The other sandwich map is handled with the legs exchanged. □

Corollary 6.3 (Simultaneous rank-four degeneracy). *Suppose $d \equiv 2 \pmod 4$ and $\mathrm{OSR}(H) = 4$. Then both maps in Theorem 6.2 are noninjective. More precisely, in the Hermitian Schmidt gauge (17), each kernel contains a nonzero tensor whose 4×4 coefficient matrix is Hermitian and traceless.*

Proof. Neither $\mathcal{O}_{\mathcal{A}}$ nor $\mathcal{O}_{\mathcal{B}}$ can be an algebra, since the five-dimensional C^*-algebra argument in the preceding proof would again give a rank-one leg projection and contradict $d \equiv 2 \pmod 4$.

The quotient-product span generated by $[A_i A_k]_{\mathcal{A}}$ is therefore nonzero and is closed under adjoint. Choose a nonzero functional on this span satisfying $\varphi(z^*) = \overline{\varphi(z)}$, and extend it to $M_d/\mathcal{O}_{\mathcal{A}}$ with the same star-reality property by extending a real basis of the self-adjoint part. Then

$$C_{ik} = \sigma_i \sigma_k \varphi([A_i A_k]_{\mathcal{A}})$$

is a nonzero Hermitian coefficient matrix, and (18) says

$$\sum_{i,k} C_{ik} B_i x B_k = 0 \quad (x \in \mathcal{O}_{\mathcal{A}}).$$

At $x = \mathbf{1}$, take ordinary trace. Hilbert–Schmidt orthonormality of the B_i gives

$$0 = \sum_{i,k} C_{ik} \operatorname{Tr}(B_i B_k) = \operatorname{Tr} C.$$

The symmetric argument gives the annihilator on the other leg. □

Corollary 6.4 (A low-Kraus-rank boundary map). *Under the hypotheses of Corollary 6.3, there is on each leg a unital completely positive map of Kraus rank at most three which acts as $-1/3$ on the opposite four-dimensional Schmidt support.*

Proof. Let $D = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_4^2) > 0$, and let $C = C^* \neq 0$ be a traceless annihilator from the preceding proof. Move from D along the line $D + tC$ to a nonzero boundary point

$$G = D + t_* C \geq 0, \qquad 1 \leq \operatorname{rank} G \leq 3.$$

The endpoint cannot be zero, since that would make C proportional to the positive matrix D, contradicting $\operatorname{Tr} C = 0$. Define

$$\Psi_B(x) = \frac{1}{d} \sum_{i,k} G_{ik} B_i x B_k.$$

Positivity of G makes this completely positive with Kraus rank at most three. The contraction of $H^2 = \mathbf{1}$ gives

$$\frac{1}{d} \sum_i \sigma_i^2 B_i^2 = \mathbf{1},$$

while a partial trace of the cubic gives

$$\frac{1}{d} \sum_i \sigma_i^2 B_i A_j B_i = -\frac{1}{3} A_j.$$

The C-direction vanishes on $\mathbb{C}\mathbf{1} + \mathcal{A}$, so Ψ_B has the same two actions. The other leg is identical. □

Remark 6.5. The map in Corollary 6.4 is not automatically trace preserving. That would require the transpose coefficient direction to annihilate the identity as well. No such transpose invariance is assumed or proved.

7 Square inheritance and the dimension-six leg intersection

The following four-strand calculation makes balance hereditary under a square restriction.

Lemma 7.1 (The two one-dimensional four-strand traces)**.** *Let μ_η be the Hecke Markov trace with $\mu_\eta(e_i) = \eta$, at $q = e^{i\pi/3}$. If e_+ and e_- are the four-strand q-symmetrizer and q-antisymmetrizer, then*

$$\begin{aligned}\mu_\eta(e_+) &= \frac{(1-\eta)(2-3\eta)(1-2\eta)}{2},\\ \mu_\eta(e_-) &= \frac{\eta(3\eta-1)(2\eta-1)}{2}.\end{aligned} \tag{20}$$

Their unique common zero is $\eta = 1/2$.

Proof. Use braid generators satisfying

$$(g_i - q)(g_i + 1) = 0, \qquad g_i^2 = (q-1)g_i + q.$$

For $w \in S_4$, let g_w be a reduced product and $\ell(w)$ its length. The two idempotents are

$$e_+ = \frac{\sum_{w\in S_4} g_w}{\sum_{w\in S_4} q^{\ell(w)}}, \qquad e_- = \frac{\sum_{w\in S_4} (-q^{-1})^{\ell(w)} g_w}{\sum_{w\in S_4} q^{-\ell(w)}}.$$

Their denominators are nonzero at the sixth root used here. Put

$$z = \mu_\eta(g_i) = q - (1+q)\eta.$$

Reducing the 24 terms by the quadratic relation, cyclicity, and the Markov rule gives

$$\mu_\eta(e_+) = \frac{(z+1)((q+1)z+1)((q^2+q+1)z+1)}{(q+1)^2(q^2+1)(q^2+q+1)},$$

$$\mu_\eta(e_-) = -\frac{(z-q)((q+1)z-q^2)((q^2+q+1)z-q^3)}{(q+1)^2(q^2+1)(q^2+q+1)}.$$

Substitute $z = q - (1+q)\eta$ and use $q^2 - q + 1 = 0$. This gives (20). The common-zero assertion is immediate from the displayed factors. The finite reduction is also replayed exactly by the accompanying verifier. □

Theorem 7.2 (Square-restriction inheritance)**.** *If $0 \neq W \subseteq V$ is square-invariant, then $\dim W$ is even and the restricted operator satisfies*

$$R|_{W\otimes W} \in \left[e^{i\pi/3}, \frac{1}{2}, \dim W\right].$$

In particular, the restriction is non-scalar.

Proof. Square invariance makes $W^{\otimes n}$ invariant under every local generator, at every n. Proposition 3.3 and Lemma 7.1 show that the ambient representation kills both e_+ and e_-, because their images are orthogonal projections of trace zero. Their restrictions to $W^{\otimes 4}$ also vanish.

Put $r = \dim W$ and let η_W be the normalized negative spectral rank of $R|_{W\otimes W}$, allowing temporarily the scalar values zero and one. The restricted unitary again has no opposite eigenvalue pair, so its normalized tensor trace is Markov. Applying (20) to the two zero restricted idempotents makes η_W a common zero of the displayed polynomials. Hence $\eta_W = 1/2$. Its negative spectral rank is $r^2/2$, so r is even. □

Corollary 7.3 (Restrictable descent). *If a solution in dimension $d \equiv 2 \pmod 4$ is restrictable, its two diagonal restrictions are balanced exceptional solutions of smaller even dimensions, exactly one of which is again congruent to $2 \pmod 4$. Consequently a least-dimensional solution in the unresolved congruence class is nonrestrictable, and no dimension-six solution is restrictable.*

Proof. Apply Theorem 7.2 to W and $W^\perp$. Their even dimensions sum to 2 mod 4, so exactly one is 2 mod 4. For $d = 6$, a proper even split is $2 + 4$, but the dimension-two exceptional class is empty [6]. □

Remark 7.4 (The one-sided gap). Theorem 7.2 rules out a two-dimensional square-invariant subspace in $d = 6$, since it would inherit the empty dimension-two class. It does not rule out a four-dimensional square-invariant W. Corollary 7.3 says only that, if such W exists, $(W^\perp)^{\otimes 2}$ is not invariant.

We now identify the precise common-leg algebra in dimension six.

Lemma 7.5 (No common rank-two projection). *If $d = 6$, there is no rank-two projection in $\mathcal{C}_L(P) \cap \mathcal{C}_R(P)$.*

Proof. Suppose z is such a projection and put $W = zV$, so $\dim W = 2$. Both tensor legs reduce H, and

$$K = H|_{W \otimes W}$$

is a Hermitian involution satisfying the exceptional cubic on $W^{\otimes 3}$. Let $Q = (\mathbf{1} - K)/2$ and $r = \operatorname{rank} Q$.

Ranks $r = 1, 3$ are impossible by a short determinant gap. In the rank-one case write

$$Q = |\operatorname{vec} S\rangle\langle \operatorname{vec} S|, \qquad \operatorname{Tr}(S^*S) = 1.$$

On $\operatorname{ran} Q_{12}$, the two-projection relation makes the compression $Q_{12}Q_{23}Q_{12}$ have eigenvalues in $\{1, 1/3\}$. Its determinant is therefore at least $1/9$. Vectorization computes the same determinant as $|\det S|^4$, while the arithmetic–geometric mean inequality gives

$$|\det S|^4 \le \frac{1}{16} < \frac{1}{9}.$$

Complementation excludes rank three. Rank two would be a member of the empty balanced dimension-two class. Hence $r = 0$ or 4, and

$$K = \varepsilon \mathbf{1}_{W \otimes W}, \qquad \varepsilon \in \{\pm 1\}.$$

Now restrict the cubic to $W \otimes W \otimes V$. This space is invariant because z lies in both leg commutants. The first adjacent copy is $X = \varepsilon \mathbf{1}$; call the second Y. Since X is scalar and $Y^2 = \mathbf{1}$,

$$XYX - YXY = Y - X.$$

The exceptional cubic instead makes this $(X - Y)/3$, so $Y = X$. Therefore

$$H|_{W \otimes V} = \varepsilon \mathbf{1}_{W \otimes V}.$$

Taking the second partial trace contradicts Corollary 3.2:

$$0 = z(\operatorname{Tr}_2 H)z = 6\varepsilon \mathbf{1}_W.$$

□

Theorem 7.6 (Scalar intersection of the two leg commutants)**.** *Every dimension-six exceptional projection satisfies*

$$\boxed{\mathcal{C}_L(P) \cap \mathcal{C}_R(P) = \mathbb{C}\mathbf{1}_6.}$$

Proof. A non-scalar finite-dimensional C^*-subalgebra of M_6 contains a nontrivial projection. Theorem 5.2 makes its rank even, hence two or four. In the latter case the complement has rank two. Both alternatives contradict Lemma 7.5. □

Remark 7.7. Theorem 7.6 concerns the intersection after identifying the two local copies of M_6. It does not prove $\mathcal{C}_L(P) = \mathbb{C}\mathbf{1}_6$ or $\mathcal{C}_R(P) = \mathbb{C}\mathbf{1}_6$ separately. Exact relative-position models show that two non-scalar even-atom subalgebras of M_6 may have scalar intersection, so this distinction is genuine.

8 Primitive-Weyl Bell and four-product Clifford-frame classes

We record two invariant model classes which are broad enough to include many natural roots-of-unity constructions, but which still force four-divisibility.

8.1 Primitive-Weyl Bell-diagonal reflections

Let

$$X|j\rangle = |j+1\rangle, \qquad Z|j\rangle = \zeta^j|j\rangle, \qquad \zeta = e^{2\pi i/d},$$

and define the generalized Bell basis

$$|\Phi_{a,b}\rangle = \frac{1}{\sqrt{d}} \sum_{j\in\mathbb{Z}_d} \zeta^{bj}|j, j+a\rangle.$$

The pair X, Z is primitive: $ZX = \zeta XZ$, and the displayed Bell lines are the simple joint eigenspaces of $X \otimes X$ and $Z^{-1} \otimes Z$.

Theorem 8.1 (Bell-diagonal divisibility)**.** *Let*

$$H = \sum_{a,b\in\mathbb{Z}_d} h_{a,b}|\Phi_{a,b}\rangle\langle\Phi_{a,b}|, \qquad h_{a,b} \in \{\pm 1\}, \quad \sum_{a,b} h_{a,b} = 0.$$

If H satisfies the exceptional cubic, then $4 \mid d$.

Proof. Each Bell line is maximally entangled, so the balance condition gives $\mathrm{Tr}_1 H = \mathrm{Tr}_2 H = 0$. Put

$$S = H_{12}, \qquad T = H_{23}, \qquad U = ST.$$

The operator U commutes with

$$\mathsf{X} = X_1X_2X_3, \qquad \mathsf{Z} = Z_1^{-1}Z_2Z_3^{-1}.$$

These obey

$$\mathsf{X}\mathsf{Z} = \zeta\,\mathsf{Z}\mathsf{X}, \qquad \mathsf{X}^d = \mathsf{Z}^d = \mathbf{1}.$$

Primitivity implies that their generated algebra is isomorphic to $M_d(\mathbb{C})$. One direct check is that the d^2 monomials $\mathsf{X}^r\mathsf{Z}^s$ are trace orthogonal. Its representation on the d^3-dimensional three-site space is therefore a direct sum of d^2 copies of the unique d-dimensional simple module. Since U commutes with this algebra, every eigenvalue multiplicity of U is divisible by d.

Using $S^2 = T^2 = \mathbf{1}$, the exceptional cubic gives

$$(U - \mathbf{1})\left(U^2 + \frac{2}{3}U + \mathbf{1}\right) = 0.$$

The roots are

$$1, \qquad \lambda_\pm = -\frac{1}{3} \pm \frac{2\sqrt{2}}{3}i.$$

Conjugation by S sends U to U^{-1}, so the λ_+- and λ_--multiplicities agree. Call the three multiplicities m_1, m, m. The zero marginals give

$$\mathrm{Tr}(U) = \mathrm{Tr}(H_{12}H_{23}) = 0.$$

Since $\lambda_+ + \lambda_- = -2/3$,

$$m_1 - \frac{2}{3}m = 0, \qquad m_1 + 2m = d^3.$$

Hence

$$m_1 = \frac{d^3}{4}, \qquad m = \frac{3d^3}{8}.$$

Divisibility of m by d gives $8 \mid 3d^2$, and therefore $4 \mid d$. □

Corollary 8.2. *The complete primitive-Weyl Bell-diagonal branch is empty in dimension six.*

Remark 8.3. Primitivity is essential: it is what makes the common three-site Weyl algebra a full matrix algebra. The theorem does not assert that an arbitrary exceptional reflection admits Bell stabilizers. A separate exact enumeration shows that all 12,870 balanced sign tables also fail in the fixed $d = 4$ Bell basis; that finite calibration is recorded in the supplement rather than used here.

8.2 Four-product Clifford frames

Theorem 8.4 (Clifford-frame divisibility)**.** *Let d be even and suppose*

$$H = \sum_{j=1}^{4} c_j A_j \otimes B_j, \qquad c_j \in \mathbb{R} \setminus \{0\},$$

where every A_j, B_j is a traceless Hermitian involution and the four product involutions

$$A_j \otimes B_j$$

anticommute pairwise. Then $4 \mid d$.

Proof. For $i \neq j$, product anticommutation says

$$(A_iA_j) \otimes (B_iB_j) = -(A_jA_i) \otimes (B_jB_i).$$

Equality of nonzero simple tensors gives a scalar ϵ_{ij}. Involutivity forces $\epsilon_{ij}^2 = 1$. Thus each local pair commutes or anticommutes, and the choices on the two legs are complementary.

Let G_A, G_B be the alternating 4×4 binary matrices which record local anticommutation. If K_4 denotes the adjacency matrix of the complete four-vertex graph, then

$$G_A + G_B = K_4 \quad \text{over } \mathbb{F}_2.$$

If a family of Hermitian involutions has binary commutation matrix of rank $2r$, symplectic Gram–Schmidt produces r mutually commuting Weyl pairs. They generate $M_{2^r}(\mathbb{C})$, so $2^r \mid d$.

Suppose $d = 2s$ with s odd. Neither G_A nor G_B can have rank four. Since K_4 has binary rank four and alternating ranks are even, the rank inequality forces

$$\operatorname{rank}_{\mathbb{F}_2} G_A = \operatorname{rank}_{\mathbb{F}_2} G_B = 2.$$

We use the following elementary four-vertex fact: if an alternating binary matrix G and its complement $K_4 + G$ both have rank two, then one of the two graphs has an isolated vertex. To see this, write the six edge variables as a, b, c, d, e, f. Rank two gives

$$af + be + cd = 0,$$

and the complement gives

$$1 + (a + b + c + d + e + f) + (af + be + cd) = 0.$$

Thus G has an odd number of edges. A three-edge graph without an isolated vertex is a path or a star; the path has nonzero Pfaffian and the star has an isolated vertex in the complement. A five-edge graph has isolated vertices in its complement.

Let U_0 be an isolated generator on the resulting leg. The same rank-two graph contains an anticommuting pair C, D. On $\mathbb{C}^{2s}$, such a pair is unitarily equivalent to

$$C = Z \otimes \mathbf{1}_s, \qquad D = X \otimes \mathbf{1}_s.$$

Since U_0 commutes with both, $U_0 = \mathbf{1}_2 \otimes L$ for a Hermitian involution $L \in M_s$. But s is odd, so $\operatorname{Tr} L \neq 0$, contradicting the assumed tracelessness of U_0. □

Remark 8.5. The load-bearing hypothesis is pairwise anticommutation of the four product terms. The theorem does not say that either local family is pairwise anticommuting, nor does it classify arbitrary operator-Schmidt-rank-four involutions. Their Schmidt factors need not be involutions and involutivity may result from cancellations among nonzero cross anticommutators.

9 The surviving dimension-six branch and open spectrum

Combining the preceding results gives the promised exact frontier.

Corollary 9.1 (Exact dimension-six frontier). *Let P be an exceptional projection on $\mathbb{C}^6 \otimes \mathbb{C}^6$, let $H = \mathbf{1} - 2P$, and let R be given by* (3)*. Then:*

(i)
$$\mathrm{OSR}(H) \geq 4, \qquad \mathrm{OSR}(R) \geq 5.$$

If $\mathrm{OSR}(H) = 4$, both intrinsic joint-sandwich maps in Theorem 6.2 are singular and have nonzero Hermitian traceless coefficient-matrix annihilators.

(ii) *R is nonrestrictable. It has no two-dimensional square-invariant local subspace.*

(iii)
$$\mathcal{C}_L(P) \cap \mathcal{C}_R(P) = \mathbb{C}\mathbf{1}_6.$$

(iv) *H is not diagonal in a primitive generalized Bell basis and is not a four-product Clifford-frame reflection of Theorem 8.4.*

Proof. Part (i) is Theorem 5.5, (5), and Corollary 6.3. Part (ii) is Theorem 7.2 and Corollary 7.3. Part (iii) is Theorem 7.6. Part (iv) follows from Theorems 8.1 and 8.4. □

Remark 9.2 (What "primitive" means here). Corollary 9.1 does not assert irreducibility in every tensor-local sense. It leaves open a four-dimensional square-invariant subspace W, provided that the complementary square $(W^{\perp})^{\otimes 2}$ is not invariant. It also leaves open non-scalar individual algebras $\mathcal{C}_L(P)$ and $\mathcal{C}_R(P)$ with scalar intersection. Thus the safe summary is that a dimension-six solution would be *nonrestrictable* and would have no common one-leg symmetry, in the precise senses of Definition 2.1 and (6).

Supplementary bounded reductions. The supplement records an additional exact $d = 4$ sitewise-unitary orbit, which is not a new class under Lechner's equivalence, and the exact color-block reduction for extensions with a four-dimensional invariant square. The latter retains genuine leakage blocks and yields no nonexistence theorem. Neither result is used in the main theorem package.

9.1 Dimension table and open problem

The exact status after this work is:

local dimension	status	reason
d odd	empty	$d^2/2 \notin \mathbb{Z}$
2	empty	Lechner's dimension-two exclusion
$4m,\ m \geq 1$	nonempty	$d = 4$ witness and identity stabilization
$4m + 2,\ m \geq 1$	open	no universal obstruction or exact witness

Open problem 9.3 (Exceptional dimension spectrum). Determine whether

$$\left[e^{i\pi/3}, \frac{1}{2}, d\right] \neq \varnothing \quad \iff \quad 4 \mid d,$$

or construct an exact solution in some dimension $d \equiv 2 \pmod 4$, beginning with $d = 6$.

Theorem 1.1 says that the abstract Jones–Wenzl tower cannot decide Problem 9.3 through simple multiplicities, central ranks, Markov weights, or their represented branching recurrences. Corollary 9.1 identifies what a first counterexample must avoid. The rank-four sandwich identities are the only result here which applies to unrestricted Schmidt supports at the first surviving rank; they leave open the simultaneous-degeneracy branch.

10 Exact verification, limitations, and disclosure

All theorem statements have human-readable proofs above. Computation is used only to replay finite algebraic reductions, calibrate formulas, and document limitation models. The public repository contains:

(i) a theorem-dependency graph separating imported results, proved lemmas, and exact computational certificates;

(ii) a verifier manifest specifying the claim, arithmetic domain, and scope of each program;

(iii) a one-command exact verifier suite for the central results;

(iv) a supplementary catalogue separating exact ansatz theorems from numerical evidence and failed construction mechanisms;

(v) seeds, commands, dependency information, raw outputs, and claim statuses for every new numerical experiment.

The central exact programs independently replay the Hecke multiplicities, two-projection block arithmetic, invariant-leg divisibility, the low-Schmidt contraction identities, finite calibrations of the unrestricted rank-four quotient and sandwich identities, the four-strand restriction polynomials, the dimension-six commutant block analysis, primitive-Weyl coefficient formulas, and the four-product Clifford parity calculation. They are checks of finite identities, not substitutes for the logical implications proved here.

The original numerical search which discovered the published $d = 4$ five-Pauli support was not preserved and is not described as reproducible. Every computation performed for the present paper has a retained source, command, seed when applicable, raw result, and commit history.

Priority and limitations

Targeted searches covered unitary Hecke tensor-space representations, controlled bipartite unitaries of Schmidt ranks two and three, generalized Yang–Baxter operators, quaternionic and Clifford realizations, Ocneanu cell systems, and the operator-algebraic theory of Yang–Baxter endomorphisms. We found no prior statement of the automatic-standardness consequence in this exceptional matrix class, the exact all-level multiplicity limitation, the twisted-control four-divisibility theorem, or the joint-sandwich rank-four obstruction. These results should therefore be described as *apparently new* pending specialist review.

The paper does not settle Problem 9.3, exclude the simultaneously singular rank-four branch, exclude a one-sided four-dimensional square restriction in $d = 6$, classify all operator-Schmidt-rank-four unitaries, or classify all $d = 4$ solutions. Negative numerical searches and the supplementary finite ansatz exclusions are not evidence of global nonexistence.

AI-assistance disclosure

OpenAI language models were used extensively to propose proof routes, derive and check symbolic identities, write and review code, search the literature, draft prose, and conduct adversarial audits. The author directed the research program and accepts responsibility for every claim. Central results were rederived in human-readable form and, where finite algebra was involved, replayed by exact programs with retained outputs. No model residual, confidence score, or unsupported model assertion is treated as mathematical evidence. The complete work trace, including failed approaches and proof repairs, is published with the paper.

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