An exceptional four-dimensional unitary Hecke Yang–Baxter operator: a five-word Pauli–Clifford normal form

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Abstract

We give an explicit 16×16 unitary Yang–Baxter operator in the class

$$\left[e^{i\pi/3}, \frac{1}{2}, 4\right].$$

On $\mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$, its reflection has a five-word real Pauli–Clifford expansion. Four words combine into an involution M, a fifth involution E anticommutes with M, and an explicit 18-word cubic certificate restricts the reflection circle $\alpha M + \beta E$ exactly to

$$\alpha^2 = \frac{2}{3}, \qquad \beta^2 = \frac{1}{3}.$$

The spectral projection has both unnormalized partial traces $2\mathbf{1}_4$, so normalized matrix trace induces the $\eta = 1/2$ Markov trace and a faithful representation of $H_n(3,6)$ for every n. A direct dimension-three obstruction, together with the known dimension-two obstruction, proves that four is minimal. Galindo and Rowell independently obtained the same local solution through a quaternionic twisted-group-algebra construction. We identify the present matrix exactly with a local-unitary normal form of the opposite of their Family III operator and extend the comparison to all braid strands. Transporting known Family III results shows that the braid images are finite and Clifford in a fixed conjugated Pauli frame. A direct scalar enhancement and skein calculation identify the trace invariant as $2P_{\mathrm{H}}(L; i, i)$; the Galindo–Rowell metric-family algorithm then gives exact deterministic polynomial-time evaluation.

Keywords. Yang–Baxter operator; Hecke algebra; braid group representation; unitary localization; fusion category; link invariant; Clifford group.

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1 Introduction and main result

Let V be a finite-dimensional complex Hilbert space. A unitary $R \in \mathrm{End}(V \otimes V)$ is a Yang–Baxter operator when

$$(R \otimes \mathbf{1}_V)(\mathbf{1}_V \otimes R)(R \otimes \mathbf{1}_V) = (\mathbf{1}_V \otimes R)(R \otimes \mathbf{1}_V)(\mathbf{1}_V \otimes R). \tag{1}$$

Lechner [6] classified the possible characters of non-involutive unitary R-matrices with two eigenvalues, up to one exceptional family. In his normalization the two eigenvalues are -1 and q, the (-1)-spectral projection P has normalized trace

$$\eta = \frac{\operatorname{rank} P}{(\dim V)^2},$$

and the classes left unresolved by that analysis are

$$\left[e^{i\pi/3}, \frac{1}{2}, 2m\right] \qquad (m \geq 1).$$

The class in base dimension two is empty; base dimension four was therefore the smallest unresolved member of this remaining family.

The ordinary-localization framework and conjecture were introduced by Rowell and Wang [8]. The specific question here was highlighted in the case study of Galindo, Hong, and Rowell [1, §5.6]. They showed that the Jones–Wenzl representations associated with $\mathcal{C}(\mathfrak{sl}_3, 6)$ have a two-dimensional unitary quasi-localization and an 8×8 $(3,1)$-generalized localization, but no two-dimensional ordinary unitary localization. They wrote that the sequence did not appear to have an ordinary localization and noted it as a possible counterexample to the Rowell–Wang localization conjecture, restated as GHR Conjecture 1.5. Lechner isolated this family as the sole remaining case in his classification preprint and explicitly identified it with the unknown localization arising from the GHR $\mathcal{C}(\mathfrak{sl}_3, 6)$ case study [6, p. 16, discussion following Proposition 3.6]. The present paper supplies an explicit operator in that family. The dimension-four construction and localization proof are independent of Lechner's classification; the minimality argument in Section 6 uses [6, Lemma 3.1 and Theorem 3.4].

Note added: independent concurrent work. Version 1.1.0 of this manuscript, containing the explicit operator and the existence, localization, and minimum-dimension results, was publicly released on 28 July 2026.[1] Galindo and Rowell subsequently posted an independent quaternionic construction [3] and report having obtained and privately circulated it earlier. The works are treated here as independent and concurrent; no claim is made about private discovery priority. Section 8 compares the local formulas exactly, and Section 9 transfers the resulting braid and link structures to the five-word representative.

The following is the central result of this paper.

Theorem 1.1. *Let*

$$q = e^{i\pi/3} = \frac{1 + i\sqrt{3}}{2}.$$

There is an explicit unitary $R \in \mathrm{End}(\mathbb{C}^4 \otimes \mathbb{C}^4)$ *such that*

$$(R + \mathbf{1}_{16})(R - q\mathbf{1}_{16}) = 0, \qquad R_1R_2R_1 = R_2R_1R_2,$$

and each eigenvalue $-1, q$ *has multiplicity eight. With the conventions* $\mathrm{Tr}_1 = \mathrm{Tr} \otimes \mathrm{id}$ *and* $\mathrm{Tr}_2 = \mathrm{id} \otimes \mathrm{Tr}$*, its* (-1)*-spectral projection* P *satisfies*

$$\mathrm{Tr}_1(P) = \mathrm{Tr}_2(P) = 2\mathbf{1}_4.$$

Therefore

$$\left[e^{i\pi/3}, \frac{1}{2}, 4\right] \neq \varnothing,$$

and the associated tensor-space representations induce faithful embeddings

$$H_n(3, 6) \hookrightarrow \mathrm{End}\big((\mathbb{C}^4)^{\otimes n}\big) \qquad (n \geq 2).$$

Under the identification in [1, §5.6], these embeddings form an ordinary unitary localization of the $\mathcal{C}(\mathfrak{sl}_3, 6)$ *Jones–Wenzl representation sequence.*

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Corollary 1.2. *The minimum local dimension of an ordinary unitary localization of this Jones–Wenzl sequence is four.*

Theorem 1.3 (Global braid and link consequences)**.** *Put $\kappa = q - 1 = q^2 = e^{2\pi i/3}$, and let $\rho_n : B_n \to \mathrm{U}((\mathbb{C}^4)^{\otimes n})$ be generated by the adjacent copies of R. Then $(R, \mathbf{1}_4, \kappa, 2)$ is an enhanced Yang–Baxter operator, with oriented-link invariant*

$$\mathcal{J}_R(\widehat{\xi}) = \kappa^{-\operatorname{wr}(\xi)} 2^{-n} \operatorname{Tr}(\rho_n(\xi)).$$

For the normalized HOMFLYPT polynomial used below,

$$\mathcal{J}_R(L) = 2P_{\mathrm{H}}(L; i, i).$$

If $c(L)$ is the number of components and $d_2(L) = \dim_{\mathbb{F}_2} H_1(\Sigma_3(L); \mathbb{F}_2)$ for the oriented triple cyclic cover specified in Section 9, then

$$\mathcal{J}_R(L) = 2(-1)^{c(L)-1}(-2)^{d_2(L)/2}.$$

Via the all-strand equivalence of Theorem 9.4 and the Family III results cited in Section 9, every image $\rho_n(B_n)$ is finite, its generators are Clifford operators in a fixed conjugated Pauli frame, and $\mathcal{J}_R$ is exactly computable by a deterministic classical algorithm in time polynomial in the strand number and braid-word length.

The formula for R is given in Section 2; the matrix identities are proved in Sections 3 and 4. Section 5 proves the full localization claim rather than only the two-site spectral statement. The minimality argument is in Section 6. Section 7 records a generalized Yang–Baxter normal form, Section 8 gives the quaternionic factorization and concurrent comparison, and Section 9 proves Theorem 1.3. Section 10 records verification and provenance, and Section 11 discusses related work, novelty, and limitations.

2 The five-word construction

Put $V = \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$, and define the real matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Here $J = XZ = -iY$, so $\{I, X, Z, J\}$ is the real Pauli–Clifford basis of $M_2(\mathbb{C})$; every tensor word in this basis is a real signed orthogonal matrix. A word such as $ZIZZ$ denotes $Z \otimes I \otimes Z \otimes Z$ on $V \otimes V$. Define

$$\begin{aligned} H = & -\frac{1}{\sqrt{6}} ZIZZ - \frac{1}{\sqrt{6}} ZIJJ - \frac{1}{\sqrt{6}} JIZJ + \frac{1}{\sqrt{6}} JIJZ \\ & - \frac{1}{\sqrt{3}} XIXX, \end{aligned} \tag{2}$$

and

$$R = \frac{q-1}{2} \mathbf{1}_{16} + \frac{q+1}{2} H = \frac{-1 + i\sqrt{3}}{4} \mathbf{1}_{16} + \frac{3 + i\sqrt{3}}{4} H. \tag{3}$$

Equations (2)–(3) are an exact specification of all 256 matrix entries in the computational basis. The visible identity in the second qubit is not an ordinary two-site identity amplification of a 4×4

Yang–Baxter operator: the active part uses one qubit of the first four-dimensional site and both qubits of the second. Section 7 identifies it instead as a $(3,2)$-generalized operator.

The relevant multiplication table is

$$X^2 = Z^2 = I, \qquad J^2 = -I,$$

$$XZ = J = -ZX, \qquad XJ = Z = -JX, \qquad JZ = X = -ZJ.$$

Thus every product of tensor words reduces to a signed word over $\{I, X, Z, J\}$.

3 A circle of reflections

The five terms in (2) have a compact algebraic explanation. Set

$$A = ZIZZ, \quad B = ZIJJ, \quad C = JIZJ, \quad D = JIJZ, \quad E = XIXX$$

and

$$M = \frac{-A - B - C + D}{2}. \tag{4}$$

Lemma 3.1. *The operators A, B, C, D, E are real symmetric involutions. The first four commute, $D = ABC$, and E anticommutes with each of A, B, C, D. Consequently*

$$M^2 = E^2 = \mathbf{1}_{16}, \qquad ME = -EM.$$

For real α, β,

$$H(\alpha, \beta) = \alpha M + \beta E$$

satisfies

$$H(\alpha, \beta)^2 = (\alpha^2 + \beta^2)\mathbf{1}_{16}.$$

Proof. The word multiplication table gives every stated commutation relation and $D = ABC$. One may also prove $M^2 = \mathbf{1}$ without expanding. Simultaneously diagonalize A, B, C, and let their joint eigenvalues be $x, y, z \in \{\pm 1\}$. Since $D = ABC$, the eigenvalue of M is

$$\frac{-x - y - z + xyz}{2} \in \{\pm 1\}.$$

The anticommutation $ME = -EM$ then makes the cross term in $(\alpha M + \beta E)^2$ vanish. □

Remark 3.2 (Graph-phase form). If $x = (-1)^a, y = (-1)^b, z = (-1)^c$, then

$$\frac{-x - y - z + xyz}{2} = -(-1)^{ab+ac+bc}.$$

Thus in the joint stabilizer eigenbasis, M is the negative phase associated with the triangle graph, while E flips all three syndrome bits. This explains the four-plus-one support in (2); no graph-state formalism is needed in the proof.

4 The cubic certificate and Yang–Baxter equation

For any $K \in \mathrm{End}(V \otimes V)$, write

$$K_1 = K \otimes \mathbf{1}_V, \qquad K_2 = \mathbf{1}_V \otimes K.$$

The Yang–Baxter equation for (3) will follow from a cubic identity for H.

Proposition 4.1. *On the unit circle $\alpha^2 + \beta^2 = 1$,*

$$H(\alpha,\beta)_1 H(\alpha,\beta)_2 H(\alpha,\beta)_1 - H(\alpha,\beta)_2 H(\alpha,\beta)_1 H(\alpha,\beta)_2 = \frac{1}{3}\big(H(\alpha,\beta)_1 - H(\alpha,\beta)_2\big) \tag{5}$$

if and only if

$$\beta^2 = \frac{1}{3}, \qquad \alpha^2 = \frac{2}{3}.$$

In particular, the four signed points with these squared coordinates are exactly the solutions on this reflection circle.

Proof. Reduce the left side of (5), minus its right side, in the six-letter word basis. Exactly the following 18 words survive:

coefficient	words
$-\alpha(3\beta^2 - 1)/6$	$IIJIJZ$, $JIZJII$, $ZIJJII$, $ZIZZII$
$+\alpha(3\beta^2 - 1)/6$	$IIJIZJ$, $IIZIJJ$, $IIZIZZ$, $JIJZII$
$-\beta(3\alpha^2 - 3\beta^2 - 1)/3$	$IIXIXX$
$+\beta(3\alpha^2 - 3\beta^2 - 1)/3$	$XIXXII$
$+\alpha(\alpha^2 - 2\beta^2)/2$	$JIJZXX$, $JIZJXX$, $XIJXZJ$, $ZIZZXX$
$-\alpha(\alpha^2 - 2\beta^2)/2$	$XIJXJZ$, $XIZXJJ$, $XIZXZZ$, $ZIJJXX$

This is a finite certificate: each row follows by repeated use of the displayed 2×2 multiplication table. Since $\{I, X, Z, J\}$ is a basis of $M_2(\mathbb{C})$, its sixfold tensor words form a basis of $\mathrm{End}((\mathbb{C}^2)^{\otimes 6})$. Hence the residual vanishes if and only if every displayed coefficient vanishes. On $\alpha^2 + \beta^2 = 1$,

$$\alpha^2 - 2\beta^2 = 1 - 3\beta^2, \qquad 3\alpha^2 - 3\beta^2 - 1 = 2(1 - 3\beta^2).$$

Hence all coefficients vanish when $3\beta^2 = 1$. Conversely, if $\alpha = 0$, the coefficient of $IIXIXX$ is $4\beta/3 \neq 0$. If $\beta = 0$, the coefficient of $IIJIJZ$ is $\alpha/6 \neq 0$. If $\alpha\beta \neq 0$, the first coefficient group forces $3\beta^2 = 1$, and the unit-circle condition then gives $\alpha^2 = 2/3$. The claim follows. $\square$

Choosing

$$\alpha = \sqrt{\frac{2}{3}}, \qquad \beta = -\frac{1}{\sqrt{3}}$$

in $H(\alpha,\beta)$ gives exactly (2). Lemma 3.1 and Proposition 4.1 therefore give

$$H^* = H, \qquad H^2 = \mathbf{1}_{16}, \qquad H_1 H_2 H_1 - H_2 H_1 H_2 = \frac{1}{3}(H_1 - H_2). \tag{6}$$

Every word in (2) contains a traceless factor, so $\mathrm{Tr}\, H = 0$. The eigenvalues $+1$ and -1 thus both have multiplicity eight.

Proof of the matrix part of Theorem 1.1. Put

$$a = \frac{q-1}{2}, \qquad b = \frac{q+1}{2},$$

so $R = a\mathbf{1} + bH$. Since $H_i^2 = \mathbf{1}$, direct expansion gives

$$R_1R_2R_1 - R_2R_1R_2 = b\left[a^2(H_1 - H_2) + b^2(H_1H_2H_1 - H_2H_1H_2)\right].$$

But

$$\frac{a}{b} = \frac{q-1}{q+1} = \frac{i}{\sqrt{3}},$$

so $a^2 + b^2/3 = 0$. Equation (6) proves the Yang–Baxter equation.

On the $+1$-eigenspace of H, $R = a + b = q$; on the -1-eigenspace, $R = a - b = -1$. Therefore

$$(R + \mathbf{1})(R - q\mathbf{1}) = 0, \qquad R^*R = \mathbf{1},$$

with both eigenvalues of multiplicity eight. The (-1)-spectral projection is

$$P = \frac{\mathbf{1}_{16} - H}{2},$$

so $P = P^* = P^2$, $\operatorname{rank} P = 8$, and $\eta = 8/16 = 1/2$. Finally, each word in H has a traceless factor on each four-dimensional site. Hence

$$\operatorname{Tr}_1(H) = \operatorname{Tr}_2(H) = 0, \qquad \operatorname{Tr}_1(P) = \operatorname{Tr}_2(P) = 2\mathbf{1}_4.$$

□

5 The full localization statement

Galindo–Rowell also establish strict localization in local dimension four [3, §13]. This section and Section 6 retain the independent direct proofs in public version 1.1.0, including scalar partial traces, the kernel–annihilator equality, and the explicit dimension-three obstruction.

We now verify that the construction localizes the whole Jones–Wenzl tower, not merely its two-site generator. Let $H_n(q)$ be the specialized Hecke algebra of type A_{n-1}, generated by $g_1, \ldots, g_{n-1}$, with the braid and far-commutativity relations and

$$(g_i + \mathbf{1})(g_i - q\mathbf{1}) = 0.$$

Its idempotent generators e_i are related to the braid generators by

$$g_i = q\mathbf{1} - (1+q)e_i.$$

The Yang–Baxter and Hecke relations define a representation

$$\rho_n : H_n(q) \longrightarrow \operatorname{End}(V^{\otimes n}), \qquad \rho_n(g_i) = R_i, \qquad \rho_n(e_i) = P_i.$$

Let

$$\tau_n(T) = 4^{-n}\operatorname{Tr}(T)$$

be normalized matrix trace.

Equip $H_n(q)$ with the conjugate-linear involution

$$e_i^* = e_i, \qquad \text{equivalently} \qquad g_i^* = g_i^{-1}.$$

Since $P_i = P_i^*$ and R_i is unitary, each ρ_n is a unital $*$-representation. If $\iota_n : H_n(q) \to H_{n+1}(q)$ is the standard inclusion, then

$$\rho_{n+1}(\iota_n(x)) = \rho_n(x) \otimes \mathbf{1}_4. \tag{7}$$

For the $\eta = 1/2$ Markov trace, write

$$\mathrm{Ann}_n = \left\{ x \in H_n(q) : \mathrm{tr}_{1/2}^{(n)}(yx) = 0 \text{ for every } y \in H_n(q) \right\}.$$

Proposition 5.1. *The traces $\tau_n \circ \rho_n$ are the Markov traces with parameter $\eta = 1/2$:*

$$\tau_{n+1}((\rho_n(x) \otimes \mathbf{1}_4)P_n) = \frac{1}{2}\,\tau_n(\rho_n(x)) \qquad (x \in H_n(q)). \tag{8}$$

For every n,

$$\ker \rho_n = \mathrm{Ann}_n .$$

Consequently ρ_n induces an injective representation of the semisimple quotient

$$H_n(3,6) = H_n(q)/\,\mathrm{Ann}_n .$$

Proof. Taking the partial trace over the last tensor factor and using $\mathrm{Tr}_2(P) = 2\mathbf{1}_4$ gives

$$\begin{aligned} \tau_{n+1}((\rho_n(x) \otimes \mathbf{1}_4)P_n) &= 4^{-(n+1)}\,\mathrm{Tr}(\rho_n(x)\ \mathrm{Tr}_{n+1}(P_n)) \\ &= 4^{-(n+1)}\,2\,\mathrm{Tr}(\rho_n(x)) = \frac{1}{2}\tau_n(\rho_n(x)). \end{aligned}$$

Moreover,

$$\tau_{n+1}(\rho_n(x) \otimes \mathbf{1}_4) = \tau_n(\rho_n(x)),$$

so the traces are consistent under the standard inclusions. Together with normalization and traciality, this is the defining Markov property. Uniqueness of the Markov trace with prescribed $\mathrm{tr}_{1/2}^{(n)}(e_i) = 1/2$ identifies it with the trace defining $H_n(3,6)$ [10, Eq. (3.2)]. Explicitly, in the normalization used for this quotient,

$$\eta_{6,3} = \frac{1 - q^{-2}}{(1+q)(1-q^{-3})} = \frac{1}{2}, \qquad q = e^{i\pi/3}.$$

This is the Wenzl Markov parameter for $H_n(3,6)$. GHR §5.6 identifies this quotient tower with the $\mathcal{C}(\mathfrak{sl}_3, 6)$ Jones–Wenzl sequence, and Theorem 5.28 uses that identification for its generalized localization.

The equality $\tau_n \circ \rho_n = \mathrm{tr}_{1/2}^{(n)}$ now gives both kernel inclusions explicitly. If $x \in \mathrm{Ann}_n$, then

$$\begin{aligned} x \in \mathrm{Ann}_n &\Longrightarrow 0 = \mathrm{tr}_{1/2}^{(n)}(x^* x) = \tau_n(\rho_n(x)^* \rho_n(x)) \\ &\Longrightarrow \rho_n(x) = 0, \end{aligned}$$

because normalized matrix trace is faithful. Conversely, for every $y \in H_n(q)$,

$$\rho_n(x) = 0 \Longrightarrow \mathrm{tr}_{1/2}^{(n)}(yx) = \tau_n(\rho_n(y)\rho_n(x)) = 0,$$

so $x \in \mathrm{Ann}_n$. Thus the kernel is exactly the annihilator and the induced map on the quotient is injective. $\square$

Remark 5.2 (Conventions in GHR §5.6). The published text of GHR §5.6 uses two q-normalizations: its opening paragraph prints $q = e^{2\pi i/3}$, whereas the braid eigenvalues and Theorem 5.28 use $q = e^{2\pi i/6}$. We use the latter eigenvalue normalization. The proof of Theorem 5.28 also prints $\eta = (1 - q^{-2})/(1 + q^3)$, whose denominator vanishes at this value. The Wenzl parameter used in Proposition 5.1 is $\eta_{6,3} = 1/2$.

Equation (7) and Proposition 5.1 give the quotient-tower compatibility explicitly:

$$\begin{aligned} x \in \iota_n^{-1}(\mathrm{Ann}_{n+1}) &\iff \rho_{n+1}(\iota_n(x)) = 0 \\ &\iff \rho_n(x) \otimes \mathbf{1}_4 = 0 \\ &\iff \rho_n(x) = 0 \\ &\iff x \in \mathrm{Ann}_n \,. \end{aligned}$$

Thus the inclusions descend to the semisimple quotients and give the commuting square

$$\begin{tikzcd} H_n(3,6) \arrow[r, "\bar{\iota}_n"] \arrow[d, "\bar{\rho}_n"'] & H_{n+1}(3,6) \arrow[d, "\bar{\rho}_{n+1}"] \\ \mathrm{End}(V^{\otimes n}) \arrow[r, "T \mapsto T \otimes \mathbf{1}_4"] & \mathrm{End}(V^{\otimes (n+1)}). \end{tikzcd}$$

Galindo, Hong, and Rowell recall the braid-equivariant isomorphism

$$\mathbb{C}\,\rho_X(B_n) \cong \mathrm{End}_{\mathcal{C}}(X^{\otimes n}) \cong H_n(3,6)$$

for the simple tensor generator X of $\mathcal{C}(\mathfrak{sl}_3, 6)$, with $\mathrm{FPdim}(X) = 2$ [1, §5.6]. Since the braid image is the entire algebra being localized, the compatible quotient embeddings above are precisely the injective algebra maps required by [1, Definition 1.3]. Equivalently, by [1, Proposition 4.12 and Definition 4.13], they define an ordinary unitary Hilb_f-localization of the Jones–Wenzl sequence.

Corollary 5.3. *The Rowell–Wang localization conjecture [8, Conjecture 3.1, p. 601], as restated in [1, Conjecture 1.5], holds for the simple tensor generator X of $\mathcal{C}(\mathfrak{sl}_3, 6)$: one has*

$$\mathrm{FPdim}(X)^2 = 4,$$

and the associated Jones–Wenzl sequence admits an ordinary unitary localization. Hence the case singled out in [1, Remark 6.2(2)] is not a counterexample.

Proof. Theorem 1.1 supplies the asserted localization, and GHR §5.6 gives $\mathrm{FPdim}(X) = 2$. This does not contradict [1, Theorem 5.27]: that result excludes a fiber functor by showing that one would force a localization of dimension exactly $\mathrm{FPdim}(X) = 2$, whereas the present ordinary localization has local dimension four. □

6 Minimality and amplification

Proof of Corollary 1.2. A one-dimensional local space cannot carry the required two-eigenvalue generator. Galindo, Hong, and Rowell prove that no two-dimensional ordinary unitary localization exists [1, Lemma 5.26]. That lemma quotes the two-dimensional classification up to an overall scalar: a representative with spectrum $\{-\chi, \chi q\}$ is normalized by multiplication by χ^{-1}, which leaves its spectral projection and Markov parameter unchanged.

It remains to exclude local dimension three. Suppose such an ordinary unitary localization existed, with local braid operator $F \in \mathrm{End}(\mathbb{C}^3 \otimes \mathbb{C}^3)$. Since a localization intertwines the braid generators,

$$(F + \mathbf{1}_9)(F - q\mathbf{1}_9) = 0.$$

Injectivity at two strands rules out either spectral projection vanishing, so both eigenvalues -1 and q occur. Thus F belongs to one of the dimension-three Hecke classes in [6, Theorem 3.4]. At $q = e^{i\pi/3}$, the possible Markov parameters are $1/3, 1/2, 2/3$. The value $1/2$ is impossible in dimension three because it would require a projection of rank $9/2$. By [6, Lemma 3.1], the normalized matrix character of each such Hecke operator is the corresponding positive Markov trace.

We now exclude the two remaining parameters directly. In $H_3(q)$, put

$$c = \frac{q}{(1+q)^2} = \frac{1}{3}, \qquad T = e_1e_2e_1 - ce_1.$$

The projection-form Hecke relation is

$$e_1e_2e_1 - e_2e_1e_2 = c(e_1 - e_2).$$

For the Markov trace $\mathrm{tr}_\eta^{(3)}$ with $\mathrm{tr}_\eta^{(3)}(e_i) = \eta$, multiply the displayed projection-form Hecke relation on the right by e_2. Cyclicity and the Markov property give

$$\begin{aligned} \mathrm{tr}_\eta^{(3)}(e_1e_2e_1e_2) &= \mathrm{tr}_\eta^{(3)}(e_2e_1e_2) + c\,\mathrm{tr}_\eta^{(3)}(e_1e_2) - c\,\mathrm{tr}_\eta^{(3)}(e_2) \\ &= \eta^2 - c\eta + c\eta^2, \end{aligned}$$

while cyclicity and $e_1^2 = e_1$ give

$$\mathrm{tr}_\eta^{(3)}(e_1e_2e_1e_2e_1) = \mathrm{tr}_\eta^{(3)}(e_1e_2e_1e_2).$$

Consequently,

$$\begin{aligned} \mathrm{tr}_\eta^{(3)}(T^*T) &= \mathrm{tr}_\eta^{(3)}(e_1e_2e_1e_2) - 2c\eta^2 + c^2\eta \\ &= (1-c)\eta(\eta - c). \end{aligned}$$

Thus a parameter-$1/3$ candidate kills T: its normalized matrix trace is faithful on its image, while the displayed squared norm is zero. In the target $\eta = 1/2$ quotient, however,

$$\mathrm{tr}_{1/2}^{(3)}(T^*T) = \frac{1}{18} > 0,$$

so T is nonzero. In particular, the constructed $\eta = 1/2$ Hecke representation does not factor through the Temperley–Lieb quotient.

For the remaining case put $f_i = \mathbf{1} - e_i$. The complemented projections satisfy the same relation with the same c, since

$$f_1f_2f_1 - f_2f_1f_2 = -\big(e_1e_2e_1 - e_2e_1e_2\big) = c(f_1 - f_2).$$

Their Markov parameter is $1 - \eta$, because for $x \in H_n(q)$,

$$\mathrm{tr}_\eta(xf_n) = \mathrm{tr}_\eta(x) - \mathrm{tr}_\eta(xe_n) = (1-\eta)\,\mathrm{tr}_\eta(x).$$

At $\eta = 2/3$, therefore, the candidate kills

$$T^{\perp} = f_1 f_2 f_1 - c f_1,$$

whereas in the target quotient

$$\mathrm{tr}^{(3)}_{1/2}\Big((T^{\perp})^* T^{\perp}\Big) = \frac{1}{18} > 0.$$

Neither dimension-three candidate is faithful. Theorem 1.1 supplies dimension four. □

There is also an immediate infinite family. For vector spaces V, W, let $\Sigma_{23} : (V \otimes W)^{\otimes 2} \to V^{\otimes 2} \otimes W^{\otimes 2}$ exchange the middle tensor factors. For $R \in \mathrm{End}(V^{\otimes 2})$ and $S \in \mathrm{End}(W^{\otimes 2})$, define the reshuffled tensor product

$$R \boxtimes S = \Sigma_{23}^{-1}(R \otimes S)\Sigma_{23}.$$

Taking $S = \mathbf{1}_{W^{\otimes 2}}$ with $W = \mathbb{C}^m$ gives a unitary Yang–Baxter operator on the base space $V \otimes W$, of dimension $4m$ [6, §3]. Its two eigenvalues and normalized spectral rank are unchanged. Therefore

$$\left[e^{i\pi/3}, \frac{1}{2}, 4m\right] \neq \varnothing \qquad (m \geq 1). \tag{9}$$

This does not settle the remaining base dimensions $6, 10, 14, \ldots$.

7 A (3,2)-generalized normal form

The visible identity in the second qubit of every word in (2) does have a precise chain-level interpretation: after a within-site coordinate swap the chain-level structure becomes explicit. We use the generalized braid placement of [1, Eq. (3.9) and Remark 3.10(2)] and the associated tower and localization conventions of [1, Example 4.23 and Definition 4.24]. Write the i-th four-dimensional site as

$$V_i = \mathbb{C}^2_{a_i} \otimes \mathbb{C}^2_{b_i},$$

let $\Sigma_{\mathrm{q}}(u \otimes v) = v \otimes u$ be the within-site qubit swap, and put

$$U_n = \Sigma_{\mathrm{q}}^{\otimes n}.$$

In the factor order obtained after applying U_n,

$$(b_1, a_1, b_2, a_2, \ldots),$$

define

$$K = \frac{q-1}{2}\mathbf{1}_8 + \frac{q+1}{2}K_H$$

and

$$K_H = -\frac{ZZZ + ZJJ + JJZ - JZJ}{\sqrt{6}} - \frac{XXX}{\sqrt{3}}. \tag{10}$$

Let $\rho_R^{(n)}$ denote the standard braid representation generated by the adjacent copies of R.

Proposition 7.1. *The two-site operator factors after the sitewise swap as*

$$U_2 R U_2^* = \mathbf{1}_{b_1} \otimes K_{a_1, b_2, a_2}.$$

The operator $K \in \mathrm{End}((\mathbb{C}^2)^{\otimes 3})$ is unitary and satisfies the $(3,2)$-generalized Yang–Baxter equation

$$(K \otimes \mathbf{1}_4)(\mathbf{1}_4 \otimes K)(K \otimes \mathbf{1}_4) = (\mathbf{1}_4 \otimes K)(K \otimes \mathbf{1}_4)(\mathbf{1}_4 \otimes K) \tag{11}$$

on $(\mathbb{C}^2)^{\otimes 5}$, together with far commutativity. If $\rho_K^{(n)}$ denotes the resulting $(3,2)$-generalized braid representation, explicitly

$$\rho_K^{(n)}(\sigma_i) = \mathbf{1}_2^{\otimes 2(i-1)} \otimes K \otimes \mathbf{1}_2^{\otimes 2(n-i-1)},$$

then for every n and every $x \in \mathbb{C}B_n$,

$$U_n \rho_R^{(n)}(x) U_n^* = \mathbf{1}_{b_1} \otimes \rho_K^{(n)}(x).$$

Moreover, for the standard inclusion $\iota_n : \mathbb{C}B_n \hookrightarrow \mathbb{C}B_{n+1}$, the placement formula gives

$$\rho_K^{(n+1)}(\iota_n(x)) = \rho_K^{(n)}(x) \otimes \mathbf{1}_4.$$

Consequently K gives a faithful unitary $(3,2)$-localization of the tower $\{H_n(3,6)\}_{n\geq 2}$.

Proof. Conjugating the five words in (2) by U_2 puts the identity factor first and leaves exactly (10) on (a_1, b_2, a_2). This proves the factorization and unitarity. On a chain, adjacent copies of K act on triples shifted by two qubits. After factoring out the global spectator b_1, the ordinary Yang–Baxter equation (1) is precisely (11); nonadjacent triples are disjoint, so far commutativity holds. The displayed chain conjugacy follows generator by generator. The same placement formula gives the displayed compatibility with ι_n. Tensoring by a spectator identity does not change the kernel, and Proposition 5.1 proves faithfulness on $H_n(3,6)$. □

Equivalently, any $(3,2)$-generalized operator K on $W^{\otimes 3}$ blocks into an ordinary operator $\mathbf{1}_W \otimes K$ on the local space $W \otimes W$, in the displayed ordering; the ordinary braid relation is $\mathbf{1}_W$ tensored with (11). This elementary blocking observation explains why the present 8×8 active operator yields a 16×16 ordinary one.

There is an exact tensor-structural distinction from the known operator $K_{\mathrm{GHR}}^{\mathrm{gen}}$ in [1, Eq. (5.2), Theorem 5.28, and Remark 5.29(1)]. For an operator L on three qubits, define

$$\begin{aligned}\mathcal{Y}_m(L) = {}&(L \otimes \mathbf{1}_{2^m})(\mathbf{1}_{2^m} \otimes L)(L \otimes \mathbf{1}_{2^m})\\ &- (\mathbf{1}_{2^m} \otimes L)(L \otimes \mathbf{1}_{2^m})(\mathbf{1}_{2^m} \otimes L).\end{aligned}$$

Direct exact calculation gives, with $\|\cdot\|_F$ the Frobenius norm,

	$\|\mathcal{Y}_1(\,\cdot\,)\|_F^2$	$\|\mathcal{Y}_2(\,\cdot\,)\|_F^2$
$K_{\mathrm{GHR}}^{\mathrm{gen}}$	0	48
K	24	0

The two nonzero values occur in different ambient matrix dimensions; only their vanishing or nonvanishing is used here. Thus the displayed GHR operator is of type $(3,1)$, while the present operator is of type $(3,2)$. The distinction concerns tensor placement, not the two matrices as bare normal operators: they have the same two eigenvalues with the same multiplicities and hence are unitarily similar after forgetting tensor structure. The $(3,2)$ shift is what permits the spectator blocking $\mathbf{1}_2 \otimes K$ into an ordinary operator on the local space $\mathbb{C}^2 \otimes \mathbb{C}^2$; the GHR operator does not yield this ordinary blocking in its displayed $(3,1)$ tensor structure.

GHR also records a different $(3,2)$ operator in Example 3.14. Its two eigenvalues have projective ratio i (up to inversion), whereas the ratio for K is $-q = e^{-2\pi i/3}$ (up to inversion), so it is a different spectral class. In particular, being of type $(3,2)$ is not by itself the distinctive feature here; the relevant structure is the exceptional Hecke spectrum together with a faithful localization of the $H_n(3,6)$ Jones–Wenzl tower.

8 Quaternionic factorization and concurrent comparison

The Pauli–Clifford reflection admits an intrinsic quaternionic factorization. This both explains its relation to Family III in the concurrent work of Galindo and Rowell and gives an exact bridge between the two explicit local formulas. To distinguish the operator in this section from the eight-dimensional generalized operator $K^{\mathrm{gen}}_{\mathrm{GHR}}$ above, write $R_{\mathrm{K}} := R$.

Proposition 8.1 (Intrinsic Family III form)**.** *With M, E as in Lemma 3.1, define*

$$\mathsf{A} = -i\sqrt{2}\,M, \qquad \mathsf{B} = iE,$$

and

$$U_{\mathrm{K}} = \frac{\mathsf{A} + \mathsf{AB}}{2}, \qquad V_{\mathrm{K}} = \frac{\mathsf{A} - \mathsf{AB}}{2}.$$

Then

$$U_{\mathrm{K}}^2 = V_{\mathrm{K}}^2 = -\mathbf{1}_{16}, \qquad U_{\mathrm{K}}V_{\mathrm{K}} = -V_{\mathrm{K}}U_{\mathrm{K}} = \mathsf{B},$$

and $U_{\mathrm{K}}, V_{\mathrm{K}}$ are skew-Hermitian unitaries:

$$U_{\mathrm{K}}^\dagger = -U_{\mathrm{K}}, \qquad V_{\mathrm{K}}^\dagger = -V_{\mathrm{K}}, \qquad U_{\mathrm{K}}^\dagger U_{\mathrm{K}} = V_{\mathrm{K}}^\dagger V_{\mathrm{K}} = \mathbf{1}_{16}.$$

Moreover,

$$U_{\mathrm{K}} + V_{\mathrm{K}} + U_{\mathrm{K}}V_{\mathrm{K}} = -i\sqrt{3}\,H. \tag{12}$$

If $\zeta = e^{\pi i/6}$, so that $q = \zeta^2$, then

$$R_{\mathrm{K}} = \frac{i\zeta}{2}\left(\mathbf{1}_{16} + U_{\mathrm{K}} + V_{\mathrm{K}} + U_{\mathrm{K}}V_{\mathrm{K}}\right). \tag{13}$$

Thus the five-word representative has the abstract quaternionic Family III form used in [3, §13].

Proof. Lemma 3.1 gives

$$\mathsf{A}^2 = -2\mathbf{1}, \qquad \mathsf{B}^2 = -\mathbf{1}, \qquad \mathsf{AB} = -\mathsf{BA}.$$

Write $U_{\mathrm{K}} = \frac{1}{2}\mathsf{A}(\mathbf{1} + \mathsf{B})$ and $V_{\mathrm{K}} = \frac{1}{2}\mathsf{A}(\mathbf{1} - \mathsf{B})$. Since $(\mathbf{1} \pm \mathsf{B})\mathsf{A} = \mathsf{A}(\mathbf{1} \mp \mathsf{B})$, direct multiplication gives the two squares and $U_{\mathrm{K}}V_{\mathrm{K}} = \mathsf{B} = -V_{\mathrm{K}}U_{\mathrm{K}}$. Since M and E are Hermitian, A and B are skew-Hermitian. The displayed formulas for $U_{\mathrm{K}}, V_{\mathrm{K}}$, together with $\mathsf{AB} = -\mathsf{BA}$, show that they too are skew-Hermitian; their squares then prove unitarity. Moreover,

$$U_{\mathrm{K}} + V_{\mathrm{K}} + U_{\mathrm{K}}V_{\mathrm{K}} = \mathsf{A} + \mathsf{B} = -i\sqrt{2}\,M + iE = -i\sqrt{3}\,H,$$

where the last equality uses $H = \sqrt{2/3}\,M - E/\sqrt{3}$. Finally $i\zeta = q - 1$ and $\sqrt{3}\,\zeta = q + 1$, so substituting (12) into (13) yields Equation (3). □

We now encode the Galindo–Rowell local operator from their literal tensor placements. On $W = \mathbb{C}^2 \otimes \mathbb{C}^2$, set

$$P_Z = (I \otimes Z) \otimes (Z \otimes I), \qquad P_X = (X \otimes I) \otimes (X \otimes X),$$

$$U = iP_Z, \qquad V = iP_X, \qquad R_{\mathrm{GR}} = \frac{i\zeta}{2}\left(\mathbf{1}_{16} + U + V + UV\right). \tag{14}$$

These are the formulas in [3, §13], not the older 8×8 operator $K^{\mathrm{gen}}_{\mathrm{GHR}}$ from GHR Equation (5.2). Let $\Sigma(x \otimes y) = y \otimes x$ swap the two four-dimensional sites.

Proposition 8.2 (Exact comparison with Galindo–Rowell)**.** *Let $\varepsilon_0, \varepsilon_1$ be the standard basis of $\mathbb{C}^2$. In the ordered basis $(\varepsilon_0 \otimes \varepsilon_0, \varepsilon_0 \otimes \varepsilon_1, \varepsilon_1 \otimes \varepsilon_0, \varepsilon_1 \otimes \varepsilon_1)$ of W, the matrix*

$$S = \frac{1}{4}\begin{pmatrix} 2+\sqrt{2} & -i\sqrt{2} & i\sqrt{2} & 2-\sqrt{2} \\ \sqrt{2} & i(2+\sqrt{2}) & i(2-\sqrt{2}) & -\sqrt{2} \\ -(2-\sqrt{2}) & -i\sqrt{2} & i\sqrt{2} & -(2+\sqrt{2}) \\ -\sqrt{2} & i(2-\sqrt{2}) & i(2+\sqrt{2}) & \sqrt{2} \end{pmatrix}$$

is unitary and satisfies

$$R_{\mathrm{K}} = (S^\dagger \otimes S^\dagger)\, \Sigma R_{\mathrm{GR}} \Sigma\, (S \otimes S). \tag{15}$$

Hence R_{K} is a local-unitary normal form of the opposite operator $R_{\mathrm{GR},21} := \Sigma R_{\mathrm{GR}} \Sigma$.

Proof. Exact row inner products give $SS^\dagger = \mathbf{1}_4$, hence $S^\dagger S = \mathbf{1}_4$. For

$$\mathcal{C}(T) = (S^\dagger \otimes S^\dagger)\Sigma T \Sigma (S \otimes S),$$

substitution of the displayed matrices gives

$$\mathcal{C}(U) = U_{\mathrm{K}}, \qquad \mathcal{C}(V) = V_{\mathrm{K}}, \qquad \mathcal{C}(UV) = U_{\mathrm{K}} V_{\mathrm{K}}.$$

Applying $\mathcal{C}$ to (14) and using Proposition 8.1 proves (15). All entries in these finite products lie in $\mathbb{Q}(\sqrt{2}, \sqrt{3}, i)$; the independent exact verifier in the source package checks the three generator identities as well as the resulting operator identity. □

Equation (15) includes a site reversal. The transformation $R \mapsto \Sigma R \Sigma$ preserves the Yang–Baxter equation, so the opposite is again a Yang–Baxter operator. Galindo–Rowell's stated local-equivalence relation permits a unit scalar and a common local basis change [3, §2], but does not include this site reversal. Proposition 8.2 therefore proves local-unitary equivalence to the opposite operator $R_{\mathrm{GR},21}$. It does not determine whether a direct local-unitary equivalence to R_{GR} also exists. The two formulas represent the same underlying local solution up to opposite and local basis change. This comparison is separate from the preceding residual table for $K^{\mathrm{gen}}_{\mathrm{GHR}}$, which distinguishes the displayed $(3,1)$ and $(3,2)$ generalized tensor placements.

9 Finite braid images and the enhanced link invariant

We now pass from the exact two-site comparison to the full braid representations and to the link invariant in the normalization of the five-word matrix. Put

$$\kappa = q - 1 = q^2 = e^{2\pi i/3}.$$

Propositions 9.1–9.3 are direct calculations from the displayed five-word operator and its scalar partial traces. Theorem 9.4 promotes the local opposite comparison to all strands. Corollaries 9.5–9.7 transfer the known Family III consequences, while Corollary 9.8 records the classical Lickorish–Millett identification.

Proposition 9.1 (Scalar enhancement)**.** *The quadruple*

$$(R, \mu, \alpha, \beta) = (R, \mathbf{1}_4, \kappa, 2)$$

is an enhanced unitary Yang–Baxter operator in Turaev's convention. Its oriented-link invariant is

$$\mathcal{J}_R(\widehat{\xi}) = \kappa^{-\operatorname{wr}(\xi)} 2^{-n} \operatorname{Tr}(\rho_n(\xi)), \qquad \xi \in B_n, \tag{16}$$

where Tr *is the ordinary unnormalized trace on the* 4^n*-dimensional tensor space and* wr *is the braid exponent sum. In particular,* $\mathcal{J}_R(\bigcirc) = 2$*. Since* $\mu = \mathbf{1}_4$*, the factor* $\mu^{\otimes n}$ *in Turaev's general trace formula is* $\mathbf{1}_{4^n}$ *and is suppressed in* (16)*.*

Proof. Equations (3) and (6), together with the partial traces computed in the proof of Theorem 1.1, give

$$\mathrm{Tr}_1(R) = \mathrm{Tr}_2(R) = 2\kappa\mathbf{1}_4.$$

The spectral decomposition also gives

$$R^{-1} = \frac{q^{-1}-1}{2}\mathbf{1}_{16} + \frac{q^{-1}+1}{2}H.$$

Since $q^{-1} - 1 = \kappa^{-1}$,

$$\mathrm{Tr}_1(R^{-1}) = \mathrm{Tr}_2(R^{-1}) = 2\kappa^{-1}\mathbf{1}_4.$$

Turaev's enhancement conditions are

$$R(\mu \otimes \mu) = (\mu \otimes \mu)R,$$

$$\mathrm{Tr}_2(R(\mu \otimes \mu)) = \alpha\beta\mu, \qquad \mathrm{Tr}_2(R^{-1}(\mu \otimes \mu)) = \alpha^{-1}\beta\mu$$

[9, §§2.2–2.3]. They hold with the displayed scalars; the commutation is immediate. Turaev's trace formula and Markov theorem then give (16) [9, §§3.1–3.2, especially Theorem 3.1.2]. For the closure of the identity in B_1, this is $2^{-1}\,\mathrm{Tr}(\mathbf{1}_4) = 2$. □

Proposition 9.2 (HOMFLYPT specialization)**.** *Let* P_{H} *be the normalized HOMFLYPT polynomial in the convention*

$$aP_{\mathrm{H}}(L_+) - a^{-1}P_{\mathrm{H}}(L_-) = zP_{\mathrm{H}}(L_0), \qquad P_{\mathrm{H}}(\bigcirc) = 1.$$

Then

$$\mathcal{J}_R(L) = 2P_{\mathrm{H}}(L; i, i). \tag{17}$$

Consequently $\mathcal{J}_R$ *is invariant under mirror image, and the c-component unlink has value* 2^c*.*

Proof. Multiplying the Hecke relation by R^{-1} yields

$$R - qR^{-1} = (q-1)\mathbf{1}_{16} = \kappa\mathbf{1}_{16}. \tag{18}$$

Apply ordinary trace to braid closures that differ only by R, R^{-1}, or the identity at one crossing. Accounting for their writhes in (16) gives

$$\mathcal{J}_R(L_+) - q\kappa^{-2}\mathcal{J}_R(L_-) = \mathcal{J}_R(L_0).$$

Because $\kappa = q^2$ and $q^3 = -1$,

$$q\kappa^{-2} = q^{-3} = -1,$$

and hence

$$\mathcal{J}_R(L_+) + \mathcal{J}_R(L_-) = \mathcal{J}_R(L_0). \tag{19}$$

At $(a, z) = (i, i)$, the displayed HOMFLYPT relation is exactly the same relation. Its normalized skein characterization and $\mathcal{J}_R(\bigcirc) = 2$ prove (17). The relation (19) is symmetric under interchange of positive and negative crossings, giving mirror invariance. The unlink value follows either from the standard split-union factor or directly from (16). □

Proposition 9.3 (Local order and twist periodicity)**.** *One has*

$$R^3 = -\mathbf{1}_{16}, \qquad R^6 = \mathbf{1}_{16}.$$

If $L_m = \widehat{x\sigma_i^m y}$, with the other braid factors fixed, then

$$\mathcal{J}_R(L_{m+3}) = -\mathcal{J}_R(L_m), \qquad \mathcal{J}_R(L_{m+6}) = \mathcal{J}_R(L_m).$$

Proof. The eigenvalues of R are $-1, q$, and both have cube -1. This proves the two operator identities. Replacing m by $m+3$ multiplies the represented braid by $-\mathbf{1}$ and increases writhe by three. Since $\kappa^3 = 1$, the writhe factor is unchanged, proving the first formula; the second follows by iteration. This is periodicity within a local twist family, not an identification of the full braid image group. □

Theorem 9.4 (All-strand equivalence)**.** *Let ρ_n^{K} and ρ_n^{GR} denote the braid representations generated by R_{K} and R_{GR}, respectively, and put $S_n = S^{\otimes n}$. Let Rev_n reverse the order of the n four-dimensional tensor sites, and let $\theta_n \in \mathrm{Aut}(B_n)$ be defined by $\theta_n(\sigma_i) = \sigma_{n-i}$. Then*

$$\mathsf{Rev}_n S_n \rho_n^{\mathrm{K}}(\xi) S_n^\dagger \mathsf{Rev}_n^\dagger = \rho_n^{\mathrm{GR}}(\theta_n(\xi)) \qquad (\xi \in B_n). \tag{20}$$

Moreover, for the Garside half twist

$$\Delta_n = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots \sigma_1,$$

put

$$D_n = \rho_n^{\mathrm{GR}}(\Delta_n), \qquad \mathcal{U}_n = D_n^\dagger \mathsf{Rev}_n S_n.$$

Then

$$\mathcal{U}_n \rho_n^{\mathrm{K}}(\xi) \mathcal{U}_n^\dagger = \rho_n^{\mathrm{GR}}(\xi) \qquad (\xi \in B_n). \tag{21}$$

Thus the image groups are unitarily conjugate and ordinary traces agree on the same braid word.

Proof. Equation (15) is equivalent to

$$(S \otimes S) R_{\mathrm{K}} (S^\dagger \otimes S^\dagger) = \Sigma R_{\mathrm{GR}} \Sigma.$$

Conjugating a two-site block by the global site reversal reverses its two internal sites and moves positions $(i, i+1)$ to $(n-i, n-i+1)$. The two reversals cancel, so generator by generator

$$\mathsf{Rev}_n S_n \rho_n^{\mathrm{K}}(\sigma_i) S_n^\dagger \mathsf{Rev}_n^\dagger = \rho_n^{\mathrm{GR}}(\sigma_{n-i}).$$

This proves (20). The standard half-twist identity

$$\Delta_n \sigma_i \Delta_n^{-1} = \sigma_{n-i}$$

says $\theta_n = \mathrm{Ad}_{\Delta_n}$. Hence the right side of (20) is $D_n \rho_n^{\mathrm{GR}}(\xi) D_n^\dagger$; conjugating by $D_n^\dagger$ proves (21). The local comparison includes an opposite, but the Garside conjugation removes the reflected generator indices at the level of the complete n-strand representation. It does not establish a direct two-site local equivalence without the opposite. □

Corollary 9.5 (Finite images and a Clifford frame)**.** *For every $n \geq 2$, the group $\rho_n^{\mathrm{K}}(B_n)$ is finite. It is Clifford with respect to a fixed conjugated Pauli frame. In particular, the images are not dense in the ambient unitary groups and are not computationally universal by braiding alone in the usual density-based sense.*

Proof. In the Galindo–Rowell realization, $U = iP_Z$ and $V = iP_X$ for the Hermitian Pauli words in (14). Since $i\zeta = \kappa$,

$$R_{\mathrm{GR}} = \kappa \left(\frac{\mathbf{1}+U}{\sqrt{2}}\right)\left(\frac{\mathbf{1}+V}{\sqrt{2}}\right) = \kappa e^{i\pi P_Z/4} e^{i\pi P_X/4}.$$

Each factor is a Pauli quarter-turn and hence normalizes the Pauli group; this is the Family III Clifford structure of [3, Proposition 8.1]. If $\mathcal{P}_{2n}$ is the standard Pauli group on the $2n$ qubits, then the original generators normalize the uniformly conjugated frame

$$(\mathsf{Rev}_n S_n)^\dagger \mathcal{P}_{2n} (\mathsf{Rev}_n S_n).$$

The need for the conjugated frame is concrete. Using the standard Hermitian Pauli $Y = iJ$ and writing $XIII = X \otimes I \otimes I \otimes I$, exact conjugation gives

$$\begin{aligned} R_{\mathrm{K}}(XIII)R_{\mathrm{K}}^\dagger = \frac{1}{2\sqrt{2}}\big(&YIYY - YIYZ + YIZY - YIZZ \\ &- ZIYY + ZIYZ - ZIZY + ZIZZ\big). \end{aligned}$$

The eight Pauli words are distinct, so R_{K} does not normalize the standard computational Pauli group; its Clifford statement is specifically relative to the displayed conjugated frame. No assertion that S itself is a standard computational Clifford operator is needed. Finiteness of the quaternionic braid image is [7, Theorem 3.1]; after the finite-order scalar normalization, it also follows in the broader Family III framework from [3, Theorem 8.4]. Theorem 9.4 transfers it to ρ_n^{K}. □

Corollary 9.6 (Quaternionic tower)**.** *Under the all-strand equivalence, the localized quotient is the braid-generated subalgebra of the explicit quaternionic twisted-group-algebra tower, where s_i denotes Rowell's quaternionic braid generator:*

$$H_n(3,6) \cong \langle s_1, \ldots, s_{n-1}\rangle_{\mathrm{alg}} \subset \mathcal{A}_n(Q_8).$$

The full tower algebra has the normal basis

$$u_1^{\epsilon_1} v_1^{\delta_1} \cdots u_{n-1}^{\epsilon_{n-1}} v_{n-1}^{\delta_{n-1}}, \qquad \epsilon_i, \delta_i \in \{0,1\},$$

and

$$\dim_{\mathbb{C}} \mathcal{A}_n(Q_8) = 4^{n-1}.$$

Proof. The basis and dimension are the literal construction in [3, §13.1]. The identification of the braid-generated subalgebra with $H_n(3,6)$ is [7, Lemma 3.2, p. 176]. This does not identify $H_n(3,6)$ with the full 4^{n-1}-dimensional tower algebra. □

Corollary 9.7 (Exact classical evaluation)**.** *Given $\xi \in B_n$, the value $\mathcal{J}_R(\widehat{\xi})$ is exactly computable by a deterministic classical algorithm in time polynomial in $n + |\xi|$.*

Proof. For Family III with the fixed metric group $A = \mathbb{Z}/2\mathbb{Z}$, [3, Theorem 7.10] gives an exact deterministic algorithm, polynomial in strand number plus braid-word length, for the coefficient-trace invariant. To match normalizations, put

$$r_0 = \kappa^{-1} R_{\mathrm{GR}} = \frac{1}{2}(\mathbf{1} + U + V + UV).$$

If ε_n is their identity-coefficient functional and Φ_n the Pauli realization, then [3, Theorem 7.3] gives

$$\operatorname{Tr}\circ\Phi_n = 4^n\varepsilon_n.$$

Because $\rho_{R_{\mathrm{GR}}}(\xi) = \kappa^{\mathrm{wr}(\xi)}\Phi_n(\rho_{r_0}(\xi))$, its enhanced trace is $2^n\varepsilon_n(\rho_{r_0}(\xi))$, exactly the invariant covered by that theorem. The same-word equivalence (21) transfers the algorithm to R_{K}. This complexity statement comes from the cited algorithm, not from the exponentially growing dimension in Corollary 9.6. □

Corollary 9.8 (Triple cyclic branched cover)**.** *Let L be an oriented link with $c(L)$ components. Let $\Sigma_3(L)$ be the connected triple cyclic cover of S^3 branched over L, obtained by completing the cover of $S^3 \setminus L$ associated with the homomorphism that sends every positively oriented meridian to $1 \in \mathbb{Z}/3\mathbb{Z}$. Put*

$$d_2(L) = \dim_{\mathbb{F}_2} H_1(\Sigma_3(L); \mathbb{F}_2).$$

Then $d_2(L)$ is even and

$$\mathcal{J}_R(L) = 2(-1)^{c(L)-1}(-2)^{d_2(L)/2}. \tag{22}$$

Proof. Lickorish–Millett normalize their polynomial $P_L(\ell, m)$ by

$$P_{\bigcirc} = 1, \qquad \ell P_{L_+} + \ell^{-1}P_{L_-} + mP_{L_0} = 0.$$

Their Theorem 2, together with the parity-preserving dimension calculation in its proof, gives

$$d_2(L) \in 2\mathbb{Z}, \qquad P_L(1,1) = (-2)^{d_2(L)/2}$$

[5, Theorem 2 and pp. 353–355]. In an oriented skein triple, $c(L_+) = c(L_-)$, while $c(L_0)$ differs by one. Therefore

$$Q(L) = (-1)^{c(L)-1}P_L(1,1)$$

satisfies $Q(L_+) + Q(L_-) = Q(L_0)$ and $Q(\bigcirc) = 1$. Comparison with the normalization in Proposition 9.2 gives

$$P_{\mathrm{H}}(L; i, i) = (-1)^{c(L)-1}(-2)^{d_2(L)/2}.$$

Equation (17) proves (22). This is the classical Lickorish–Millett evaluation; the point here is its exact identification with the invariant of the displayed five-word matrix. □

For orientation and crossing conventions, the exact matrix trace gives the following small checks:

closure	$\mathcal{J}_R$
$1 \in B_1$ (unknot)	2
$1 \in B_2$ (two-component unlink)	4
$\sigma_1^{\pm 2} \in B_2$ (Hopf links)	-2
$\sigma_1^{\pm 3} \in B_2$ (trefoils)	-4
$\left(\sigma_1\sigma_2^{-1}\right)^2 \in B_3$ (figure-eight knot)	-4
$\left(\sigma_1\sigma_2^{-1}\right)^3 \in B_3$ (Borromean rings)	2

These values agree with (22); the positive and negative closures also illustrate mirror insensitivity. For the B_3 rows, σ_1 and σ_2 act by the adjacent placements $R \otimes \mathbf{1}_4$ and $\mathbf{1}_4 \otimes R$, respectively.

10 Verification and provenance

The source package supplies five supported exact-verification routes:

1. a hardened SymPy matrix verifier based on the original discovery-era checker;
2. an independently written standard-library matrix verifier over $\mathbb{Q}(\sqrt{2}, \sqrt{3}, i)$;
3. a matrix-free Pauli-word verifier for the complete 18-word certificate in Proposition 4.1;
4. an independently encoded verifier for the intrinsic quaternionic form, the literal Galindo–Rowell operator, the site swap, and S; and
5. a braid-and-link verifier that rechecks the intrinsic factorization, S-unitarity, and two-site comparison, then checks the scalar enhancement, matrix skein identity, local order, two- and three-strand link values, the standard-frame non-Clifford witness, Pauli quarter-turns, and the global reversal and Garside conjugacies at $n = 3, 4$.

Together these routes check the displayed matrix identities, partial traces, both dimension-three obstructions, the six-dimensional three-strand Hecke image, the generalized operator and far commutativity, the exact residual comparison with $K^{\mathrm{gen}}_{\mathrm{GHR}}$, and the finite identities in Sections 8–9. All comparisons are exact. Negative tests deliberately alter coefficients, tensor placements, the local unitary, the quarter-turn order, ζ, κ, the writhe factor, the skein sign, a reversed generator index, and the Garside word. The README records the commands and scope. These finite checks do not replace the printed tower-faithfulness and all-strand conjugacy proofs, Lechner's classification input, or the literature-based interpretation of the concurrent and topological results.

The five-word support was originally found by an AI-assisted structured numerical search in a real Pauli–Clifford basis. The discovery code and random seeds were not retained. Accordingly, this paper makes no claim that the search was reproducible, exhaustive, or established uniqueness. The generative-AI systems identified in the declaration below were also used for mathematical exploration and derivation, adversarial proof analysis, verifier development, literature support, and manuscript preparation. Every theorem claim is supported by the printed derivation, and the finite identities are checked by the exact programs. No conclusion depends on accepting model output as evidence or reproducing the original numerical search.

11 Related work, novelty, and limitations

Rowell and Wang introduced the localization framework and conjecture [8]. Galindo, Hong, and Rowell then studied the $\mathcal{C}(\mathfrak{sl}_3, 6)$ sequence, excluded ordinary local dimension two, and supplied quasi- and $(3, 1)$-generalized localizations, but no ordinary dimension-four operator [1, §5.6]. Rowell's quaternionic model and the unitary-braided-vector-space literature are foundational prior art [7, 2]. Lechner later identified $[e^{i\pi/3}, 1/2, 2m]$ as the remaining existence family up to braid-character equivalence and stated that its dimension-two member is empty [6, p. 16, discussion following Proposition 3.6].

Galindo and Rowell independently realized the same exceptional local class and proved faithful strict localization of $H_n(3, 6)$ in local dimension four, the smallest dimension in Lechner's exceptional family, using a quaternionic twisted-group-algebra tower [3, §13]. Their construction is substantially broader in its classification setting. The present proof architecture is different: it gives a five-word real Pauli–Clifford representative, reduces the braid relation to two anticommuting involutions, prints the complete 18-word cubic certificate, derives the Markov trace from scalar partial traces, proves the dimension-three obstruction directly, and exhibits the active $(3, 2)$-generalized normal form. Proposition 8.2 supplies an exact bridge to their Family III formula, and Theorem 9.4 extends it to every braid word.

The public and reported private chronology is recorded in the Note added in the Introduction; the papers are treated throughout as independent and concurrent. The finite-image, Clifford, twisted-tower, and classical link-evaluation phenomena are known for the quaternionic representation. The contribution here is their exact transfer to, and transparent derivation from, the sparse five-word representative, including its scalar enhancement and matrix-trace normalization. No new HOMFLYPT evaluation, branched-cover theorem, or all-n finite-image group classification is claimed. The result does not classify the exceptional family in every even dimension or prove uniqueness. The comparison theorem concerns the 16×16 ordinary Family III operator R_{GR}, not the older 8×8 generalized operator $K^{\mathrm{gen}}_{\mathrm{GHR}}$.

Data, code, and source availability

The source, exact verifiers, frozen outputs, checksums, and audit materials for this edition accompany the manuscript in a deterministic version 1.2.0 package identified by the version-specific Zenodo DOI doi:10.5281/zenodo.22013710 [4]. The preceding version 1.1.3 package remains permanently archived at doi:10.5281/zenodo.21971507; it is not overwritten by this revision. The maintained project page provides the public manuscript and verification links. No empirical dataset was generated or analyzed.

CRediT authorship contribution statement

Alec Kriebel: Conceptualization, formal analysis, investigation, methodology, software, validation, writing–original draft, and writing–review and editing.

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Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

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