# An explicit Hessian-nilpotent quartic in 54 variables witnessing the failure of Zhao's Vanishing Conjecture

Alec Kriebel
with heavy assistance from ChatGPT 5.6 Sol

First public release: 21 July 2026, 13:11:39 UTC
Superseded-status correction: 21 July 2026, 15:20:23 UTC

**Superseded status.** Exploration 03 replaces the 27-variable cubic and 54-variable quartic dimension counts by 22 and 44. Thompson had also posted a 24-variable cubic map before this note's first release. This document is retained as a timestamped derivation of the factor-reusing 13-variable stable model and its original 54-variable certificate; its compactness claim is historical, not current. Its surviving construction has been consolidated into the canonical 22/44-variable paper; this note should not be cited separately.

**Abstract**

We execute a factor-reusing Bass–Connell–Wright degree reduction of the recently announced three-dimensional noninjective Keller map, obtaining a cubic homogeneous noninjective Keller map in 27 variables. Applying the de Bondt–van den Essen symmetric reduction then gives an explicit homogeneous quartic Hessian-nilpotent polynomial in 54 variables, with 598 monomials, whose gradient Keller map is noninjective. This is an explicit witness to the failure of Zhao's Vanishing Conjecture, with the expanded polynomial and a two-point collision supplied as machine-readable exact certificates. The underlying three-dimensional map was posted by Levent Alpöge on 20 July 2026, crediting Akhil Mathew for posing the question and Claude Fable for producing the example.

**Verification status.** The algebra has passed the supplied exact symbolic checks, but this short note has not been peer reviewed. Independent expert scrutiny is welcome.

## 1 Statement of the result

Let $\mathrm{i}^2 = -1$. The straight-line construction below defines a cubic homogeneous map $h : \mathbb{C}^{27} \to \mathbb{C}^{27}$. For $A, B \in \mathbb{C}^{27}$, set

$$\boxed{\mathcal{P}(A,B) = \mathrm{i}\sum_{j=1}^{27} h_j(A+\mathrm{i}B)B_j.} \tag{1}$$

**Theorem 1.** *The polynomial $\mathcal{P}$ is homogeneous of degree four, has 598 monomials when expanded, and its Hessian is nilpotent. The polynomial map*

$$\Gamma : \mathbb{C}^{54} \longrightarrow \mathbb{C}^{54}, \qquad \Gamma(Z) = Z - \nabla\mathcal{P}(Z), \tag{2}$$

*has Jacobian determinant one and is noninjective. Moreover,*

$$\Delta^m \mathcal{P}^m = 0 \quad \textit{for every } m \geq 1, \qquad \Delta^m \mathcal{P}^{m+1} \neq 0 \quad \textit{for infinitely many } m \geq 0. \tag{3}$$

*Thus $\mathcal{P}$ is a counterexample to Zhao's Vanishing Conjecture.*

The formulas below and ordinary differentiation form a finite exact certificate. The accompanying `potential_sparse.json` is the fully expanded polynomial over $\mathbb{Q}(\mathrm{i})$, and `collision.json` contains two distinct points with the same image under (2).

## 2 A 13-variable cubic stable model

Write $u = 1 + xy$. Postcomposing the map announced in [4] by the inverse of its linear part gives the identity-linear Keller map $\Phi = (\Phi_1, \Phi_2, \Phi_3)$:

$$\begin{aligned} \Phi_1 &= x - \frac{3}{2}x^2 y - \frac{1}{2}x^3 z, \\ \Phi_2 &= y + 3xu^2 z + 3xy^2(4 + 3xy), \\ \Phi_3 &= u^3 z + y^2 u(4 + 3xy). \end{aligned} \tag{4}$$

Thus $\det J\Phi = 1$, and the three points

$$(0, 0, -\tfrac{1}{4}), \qquad (1, -\tfrac{3}{2}, \tfrac{13}{2}), \qquad (-1, \tfrac{3}{2}, \tfrac{13}{2}) \tag{5}$$

have the same image $(0, 0, -\frac{1}{4})$.

We use the following stable degree-reduction gadget. Suppose a term $cPQ$ is to be removed from the $k$th coordinate of a current map $M$. With new variables $a, b$, replace

$$M_k \longmapsto M_k - c(a + P)(b + Q), \qquad (a, b) \longmapsto (a + P, b + Q). \tag{6}$$

This is a pre- and post-composition of $M \times \mathrm{id}_{\mathbb{A}^2}$ by triangular automorphisms. A previously created output factor may be reused: with one new variable $a$, one may replace

$$M_k \longmapsto M_k - c(a + P)M_\ell, \qquad a \longmapsto a + P. \tag{7}$$

Both operations preserve the Jacobian determinant and invertibility. A collision $p \neq q$ lifts through (6) by appending $(-P(p), -Q(p))$ and $(-P(q), -Q(q))$, and similarly for (7).

Starting from (4), perform the following operations in order. The symbols $a_j, b_j$ denote the input variables introduced at each step.

| step | target | new or reused factors | $c$ |
|---|---|---|---|
| 1 | $\Phi_1$ | $P = x^2$, $Q = xz + 3y$; add $a_1, b_1$ | $-1/2$ |
| 2 | $\Phi_2$ | $P = 3x^2 y$, $Q = 2z + xyz + 3y^2$; add $a_2, b_2$ | 1 |
| 3 | $\Phi_2$ | $P = xy$, $Q = a_2 z + 3xb_2$; add $a_3, b_3$ | $-1$ |
| 4 | $\Phi_3$ | $P = xy^2$, $Q = 7y + 3xz + 3xy^2 + x^2 yz$; add $a_4, b_4$ | 1 |
| 5 | $\Phi_3$ | add $a_5 + a_4 xy$; reuse output $b_1 + xz + 3y$ | $-1$ |
| 6 | $\Phi_3$ | reuse output $a_3 + xy$; add $b_6 + a_4 b_1 - y b_4$ | 1 |

Finally postcompose by the shear

$$M_{b_4} \longmapsto M_{b_4} - M_{a_3} M_{b_1}. \tag{8}$$

The resulting map $M$ has 13 variables

$$X = (x, y, z, a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_6) \tag{9}$$

and degree three. Its linear part $L$ is the identity except for

$$M_{b_1}^{(1)} = b_1 + 3y, \qquad M_{b_2}^{(1)} = b_2 + 2z, \qquad M_{b_4}^{(1)} = b_4 + 7y. \tag{10}$$

In particular $\det L = 1$. Put

$$\Psi = L^{-1}M = X + H_2(X) + H_3(X), \tag{11}$$

where $H_d$ is homogeneous of degree $d$. Then $\deg \Psi = 3$, $\det J\Psi = 1$, and $\Psi$ has the following two preimages of the same point:

$$\begin{aligned} p_0 &= (0, 0, -\tfrac{1}{4}, 0, 0, 0, \tfrac{1}{2}, 0, 0, 0, 0, 0, 0), \\ p_1 &= (1, -\tfrac{3}{2}, \tfrac{13}{2}, -1, -2, \tfrac{9}{2}, -10, \tfrac{3}{2}, \tfrac{3}{4}, -\tfrac{9}{4}, -6, -\tfrac{27}{8}, \tfrac{9}{2}), \\ \Psi(p_0) = \Psi(p_1) &= (0, 0, -\tfrac{1}{4}, 0, 0, 0, \tfrac{1}{2}, 0, 0, 0, 0, 0, 0). \end{aligned} \tag{12}$$

Equations (4) and (6)–(11) are a straight-line specification of all components of $\Psi$; no omitted choice is involved.

## 3 The 27-variable cubic homogeneous map

For $X, Y \in \mathbb{C}^{13}$ and $t \in \mathbb{C}$, define

$$h(X, Y, t) = \big(tH_2(X) + t^2 Y, -H_3(X), 0\big). \tag{13}$$

Every component of $h$ is cubic homogeneous. Let $\mathcal{B}(W) = W + h(W)$. At $t = 1$,

$$\mathcal{B}(X, Y, 1) = \big(X + H_2(X) + Y, Y - H_3(X), 1\big). \tag{14}$$

Consequently, if

$$r_j = (p_j, H_3(p_j), 1), \tag{15}$$

then $\mathcal{B}(r_0) = \mathcal{B}(r_1) = (p_0, 0, 1)$. Here $H_3(p_0) = 0$, while

$$\begin{aligned} H_3(p_1) = (&-\tfrac{17}{4}, -\tfrac{45}{8}, \tfrac{99}{16}, 0, \tfrac{135}{8}, -\tfrac{9}{2}, -\tfrac{177}{8}, 0, 0, \tfrac{9}{4}, \\ &\tfrac{213}{8}, \tfrac{27}{8}, 0). \end{aligned} \tag{16}$$

**Lemma 2.** *The polynomial matrix $Jh$ is nilpotent.*

*Proof.* Homogeneity and (11) give

$$\det\big(I + tJH_2(X) + t^2 JH_3(X)\big) = \det J\Psi(tX) = 1. \tag{17}$$

The Jacobian determinant of $W \mapsto W + h(W)$ is the same determinant by the block determinant formula. Hence $\det(I + Jh(W)) = 1$. Since $h$ is cubic homogeneous, $Jh(sW) = s^2 Jh(W)$. Therefore $\det(I + \lambda Jh) = 1$ identically in $\lambda$, which is equivalent to nilpotence of $Jh$. □

## 4 Symmetrization and the collision

For a polynomial map $h : \mathbb{C}^r \to \mathbb{C}^r$, de Bondt and van den Essen [2] associate

$$f_h(A, B) = -\mathrm{i} \sum_{j=1}^{r} h_j(A + \mathrm{i}B)B_j. \tag{18}$$

Their characteristic-polynomial identity implies that $\mathrm{Hess}(f_h)$ is nilpotent if and only if $Jh$ is nilpotent. Our polynomial in (1) is $\mathcal{P} = -f_h$, so Lemma 2 proves that $\mathcal{P}$ is Hessian nilpotent. It follows immediately that $\det J\Gamma = 1$.

For completeness, the collision can be seen directly. With the linear map $S(x, y) = (x - \mathrm{i}y, y)$, differentiation gives

$$S^{-1}\Gamma S(x, y) = \big(x + h(x), (I + Jh(x)^T)y - \mathrm{i}h(x)\big)\,. \tag{19}$$

Take $r_0, r_1$ from (15), let $K_j = I + Jh(r_j)^T$, put

$$y_1 = 0, \qquad y_0 = K_0^{-1}\big(-\mathrm{i}h(r_1) + \mathrm{i}h(r_0)\big), \tag{20}$$

and set $z_j = S(r_j, y_j)$. Equation (19) and $r_0 + h(r_0) = r_1 + h(r_1)$ show $\Gamma(z_0) = \Gamma(z_1)$. The two 54-tuples are recorded explicitly in `collision.json`; all coordinates have height at most 261. Direct differentiation of the 598-term expansion checks this equality exactly over $\mathbb{Q}(\mathrm{i})$.

## 5 Failure of the Vanishing Conjecture

Let $\Delta = \sum_j \partial^2/\partial Z_j^2$. Zhao proved [3] that Hessian nilpotence of a homogeneous polynomial $P$ is equivalent to

$$\Delta^m P^m = 0 \qquad \text{for every } m \geq 1. \tag{21}$$

Lemma 2 and the symmetric reduction therefore show that $\mathcal{P}$ satisfies the hypothesis of the Vanishing Conjecture. Zhao's inversion formula writes the formal inverse of $Z - t\nabla P$ as $Z + t\nabla Q_t$, where

$$Q_t = \sum_{m=0}^{\infty} \frac{t^m}{2^m m!(m+1)!}\, \Delta^m P^{m+1}. \tag{22}$$

If $\Delta^m \mathcal{P}^{m+1} = 0$ for all sufficiently large $m$, then $Q_t$ is a polynomial. Specializing at $t = 1$ would give a polynomial inverse of $\Gamma$, contradicting the explicit collision above. Thus $\Delta^m \mathcal{P}^{m+1} \neq 0$ for infinitely many $m$, proving Theorem 1.

This argument does not exhibit a particular exponent $m$ for which $\Delta^m \mathcal{P}^{m+1} \neq 0$; it deduces the existence of infinitely many such exponents indirectly from the collision and the inversion formula. Zhao's general finite bound for quartics in $n$ variables reaches $m > \frac{3}{2}(3^{n-2} - 1)$, or $m > \frac{3}{2}(3^{52} - 1)$ here, so direct expansion at the general bound is impractical [3].

## 6 Reproducibility

The repository contains two complementary exact checks. The script `verify.py` rebuilds the construction symbolically and verifies the original map, all six stable reduction gadgets, the cubic homogeneous map, Hessian nilpotence through the structural reduction, and the final collision. The dependency-free script `verify_exported_stdlib.py` uses only Python's standard library and the

two JSON files to verify the expanded polynomial's degree and term count and the exact collision. The latter is not a stand-alone proof of Hessian nilpotence. All arithmetic in both checks is exact. The certificate hashes are

```
potential_sparse.json:
556683b30fd9e7b9ecfb4c4d5395ee3be6a8dc5f918da08ed60db8c125f0df28
collision.json:
cbea49f88cc59ba7955a4dbd710c68224590ae310a320369c49f8c63e1f78cae
```

The full source and certificates are available in the public project repository.

## 7 Historical scope and post-release correction

Zhang's consequence note of 20 July 2026 observes, by Zhao's equivalence, that the announced three-dimensional counterexample makes the Vanishing Conjecture false in some finite dimension; it explicitly leaves the consequence existential [5]. An earlier public repository [6] gives a 79-variable cubic homogeneous noninjective Keller map. Applying the standard symmetric reduction to that map would produce a 158-variable quartic in principle, but the repository did not supply an expanded quartic witness or its transported collision at the time of our search.

At initial release, this note presented the executed 54-variable, 598-term quartic, its exact collision, and the 27-variable factor-reusing reduction as its narrower candidate contribution. That priority language is retained here only as historical context. It is not the current assessment.

**Post-release correction.** Thompson's initial public commit [7], at 03:29:42 UTC on 21 July 2026, already gave a 24-variable cubic homogeneous reduction and therefore beat this note's 27-variable dimension headline before our 13:11:39 UTC release. We failed to find it in the first audit. Exploration 03 subsequently compressed our 13-variable stable model to a 22-variable cubic and a 44-variable quartic. Accordingly, this note is superseded for dimension comparisons or as a separate current result. Its surviving role is as the derivation of that stable model and as a reproducible earlier certificate. Both have now been incorporated into the canonical Exploration 03 paper.

## AI-assistance and verification disclosure

The construction, proof organization, verification programs, website, and typeset drafts were developed with extensive assistance from ChatGPT 5.6 Sol. Alec Kriebel takes responsibility for the submission and for preserving the complete source and exact certificates. The algebraic checks are reproducible, but the note has not been peer reviewed and independent expert scrutiny is welcome.

## References


[1] H. Bass, E. H. Connell, and D. Wright, "The Jacobian conjecture: reduction of degree and formal expansion of the inverse," *Bull. Amer. Math. Soc.* 7 (1982), 287–330. doi:10.1090/S0273-0979-1982-15032-7.

[2] M. de Bondt and A. van den Essen, "A reduction of the Jacobian conjecture to the symmetric case," *Proc. Amer. Math. Soc.* 133 (2005), 2201–2205. doi:10.1090/S0002-9939-05-07570-2.

[3] W. Zhao, "Hessian nilpotent polynomials and the Jacobian conjecture," *Trans. Amer. Math. Soc.* 359 (2007), 249–274. arXiv:math/0409534.

[4] L. Alpöge, X post announcing the three-dimensional map, crediting Akhil Mathew for posing the question and Claude Fable for producing the example, 20 July 2026. Original post.

[5] Z. Zhang, "Direct consequences of the three-dimensional counterexample to the Jacobian conjecture," 20 July 2026. Web note.

[6] A. Harrison, `jacobian-anatomy`, public GitHub repository, commit `74808fb2e1c1691b0007576ba0508e5e7cdcb1e3`, 20 July 2026. Repository commit.

[7] W. Thompson, "An explicit 24-variable cubic-homogeneous reduction of the Alpöge–Fable Jacobian counterexample," public GitHub repository, commit `45a7616fdf5a20c065564f2676190093722696b9`, 21 July 2026. Repository.

## Archival citation status

Do not cite this as a separate current result. Cite the canonical consolidated paper, "An explicit 44-variable vanishing witness from a 22-variable cubic Keller map," instead. This page remains the timestamped provenance record for the 13-variable reduction and original 54-variable certificate.