A Certified Order-Twelve Extension of the γ–θ Frontier in One-Guard Eternal Domination

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Abstract

In the one-guard-moves eternal domination game, each attack is made at an unoccupied vertex and exactly one adjacent guard moves to the attacked vertex. The γ–θ conjecture asserts that $\gamma(G) = \gamma^\infty(G)$ implies $\gamma(G) = \theta(G)$. MacGillivray, Mynhardt, and Virgile reported an exhaustive computation excluding counterexamples through order 11. We extend that published frontier by proving that no counterexample has order 12. The order-12 proof has three separately delimited parts: an independently replayed three-template certificate excludes common parameter 3; an exact 18,381-variable formula and a 228,381,671-byte LRAT refutation exclude connected graphs with common parameter 4; and a simplicial closed-neighborhood reduction, combined with the McCuaig–Shepherd domination bound, excludes common parameter 5. A classical half-order characterization proves that these are the only possible parameters. Consequently, subject explicitly to the published through-order-11 computation, every counterexample has at least 13 vertices. The campaign has not independently reproduced the all-graph coverage at orders 10 and 11. As a structural byproduct, we give an unrestricted, family-level generalization of a planar independent-antineighborhood lemma of Taletskii. This finite result neither proves the universal conjecture nor constructs a counterexample.

1 Introduction

Let G be a finite simple graph. The domination number $\gamma(G)$ is the minimum size of a set whose closed neighborhood is all of $V(G)$, and the clique cover number $\theta(G)$ is the minimum number of parts in a partition of $V(G)$ into cliques. In the standard one-guard eternal domination game, guards initially occupy a dominating set. After an attack at an unoccupied vertex, exactly one adjacent guard moves to the attacked vertex, and the new guard set must again be capable of defending every future attack. The least sufficient number of guards is $\gamma^\infty(G)$.

Klostermeyer and MacGillivray stated the implication

$$\gamma(G) = \gamma^\infty(G) \quad \Longrightarrow \quad \gamma(G) = \theta(G), \tag{1.1}$$

with a proof that was later found to be incomplete [4, 5]. Klostermeyer and Mynhardt subsequently formulated the question as the existence problem

$$\gamma(G) = \gamma^\infty(G) < \theta(G) \tag{1.2}$$

[5, 6]. We call (1.1) the γ–θ conjecture.

MacGillivray, Mynhardt, and Virgile reported an exhaustive computation finding no counterexample through order 11 [7, Observation 5.6]. Their paper also catalogs all 56 graphs of order at most 11 for which $\gamma^\infty < \theta$. The present work independently reproduced the parameters of that catalog and independently enumerated connected graphs through order 9, but it did not repeat the original cluster-scale all-graph enumeration at orders 10 and 11. We therefore keep the published lower-order computation visible as an explicit premise.

Theorem 1.1 (order-12 frontier)**.** *Assume the published exhaustive result of MacGillivray, Mynhardt, and Virgile that no counterexample to* (1.1) *has order at most* 11*. Then no finite simple graph G of order at most* 12 *satisfies*

$$\gamma(G) = \gamma^{\infty}(G) < \theta(G).$$

Equivalently, every counterexample, if one exists, has order at least 13*.*

The order-12 extension in Theorem 1.1 is assembled from the following results.

1. A complete, certificate-backed exclusion of all 12-vertex graphs with $\gamma = \gamma^{\infty} = 3 < \theta$ (Theorem 4.1).
2. A complete, certificate-backed exclusion of connected 12-vertex graphs with $\gamma = \gamma^{\infty} = 4 < \theta$ (Theorem 5.1).
3. A structural theorem for simplicial vertices (Theorem 3.1) and the McCuaig–Shepherd bound, which exclude the order-12 parameter-five case relative to the same published lower-order premise (Corollary 3.6).
4. A classical half-order characterization, which shows that a connected order-12 counterexample has common parameter only $3, 4$, or 5 (Corollary 2.12).
5. An independent-antineighborhood projection theorem for arbitrary equality graphs and arbitrary eternal families (Theorem 3.7), recorded as a generalization of Taletskii's planar minimum-counterexample lemma rather than as a wholly new antineighborhood idea.

Every computational implication is separated into a graph-to-formula coverage theorem, an exact byte binding, a proof-certificate replay, and an independent audit. Section 2 fixes the model and elementary reductions. Section 3 proves the structural reduction. Sections 4 and 5 give the two certified slices. Section 6 assembles the frontier, and Section 7 gives reproduction details.

Remark 1.2 (scope). Theorem 1.1 is a finite extension of a published computational frontier. It is not a campaign-only exhaustive enumeration through order 12, it says nothing about orders at least 13, and it does not resolve the universal conjecture. We also make no literature-novelty claim for the simplicial reduction; one potentially relevant 2018 manuscript could not be obtained during the audit. The antineighborhood theorem below has direct accessible overlap with Taletskii's Lemma 13 [10], and its prior-art boundary is stated explicitly.

2 Model and elementary reductions

All graphs in this paper are finite, simple, and undirected. For $v \in V(G)$, write $N(v)$ and $N[v]$ for its open and closed neighborhoods. A set $D \subseteq V(G)$ is dominating if $N[D] = V(G)$. Let $\alpha(G)$ denote the independence number and $i(G)$ the minimum size of a maximal independent set. We use

$$\theta(G) = \chi(\overline{G}). \tag{2.1}$$

Definition 2.1. An *eternal dominating family of size* k is a nonempty family $\mathcal{D} \subseteq \binom{V(G)}{k}$ such that:

1. every $D \in \mathcal{D}$ dominates G; and
2. for every $D \in \mathcal{D}$ and every unoccupied attacked vertex $r \in V(G) - D$, there is a guard $u \in D \cap N(r)$ for which
$$(D - \{u\}) \cup \{r\} \in \mathcal{D}. \tag{2.2}$$

The minimum such k is $\gamma^{\infty}(G)$.

Thus attacks at occupied vertices are not required, and a response in (2.2) moves exactly one guard along exactly one edge. No all-guards-move parameter is used anywhere in this paper.

Lemma 2.2 (independent-set forcing)**.** *Let $\mathcal{D}$ be an eternal dominating family of k-sets, and let I be an independent set. Starting from any state in $\mathcal{D}$, successive attacks at unoccupied vertices of I force a state D' with*
$$|D' \cap I| = \min\{|I|, k\}.$$
In particular, every independent k-set belongs to every eternal k-family.

Proof. When an unoccupied $r \in I$ is attacked, no guard already on I can respond because I is independent. Every legal response therefore increases the number of guards on I by one. Repeat until no further such attack is possible. □

Proposition 2.3 (parameter chain)**.** *For every nonempty graph G,*
$$\gamma(G) \le i(G) \le \alpha(G) \le \gamma^{\infty}(G) \le \theta(G). \tag{2.3}$$
Consequently, if $\gamma(G) = \gamma^{\infty}(G) = k$, then
$$\gamma(G) = i(G) = \alpha(G) = \gamma^{\infty}(G) = k, \tag{2.4}$$
and G is well-covered.

Proof. Every maximal independent set dominates, and every maximum independent set is maximal. If an eternal family used fewer than $\alpha(G)$ guards, Lemma 2.2 would put all guards on a maximum independent set while leaving another vertex of it unoccupied; an attack there would have no adjacent guard. Thus $\alpha \le \gamma^{\infty}$.

Given a clique partition, place one guard in each part. An attack is answered by moving the unique guard within the attacked vertex's clique. This proves $\gamma^{\infty} \le \theta$. The equality collapse and well-coveredness follow by squeezing. □

Lemma 2.4 (induced-subgraph monotonicity)**.** *If J is a nonempty induced subgraph of G, then $\gamma^{\infty}(J) \le \gamma^{\infty}(G)$.*

Proof. Fix an eternal family in G, let m be the maximum number of guards that any family state places in $V(J)$, and retain the states attaining m. An attack inside J cannot be answered by moving a guard from outside J, since the successor would contain more than m guards in J. Thus every response stays within J, preserves m, and projects to a legal response in the induced graph J. The projected states dominate J: each unoccupied vertex has an adjacent responding guard in J, while occupied vertices dominate themselves. Also $m > 0$, since otherwise any attack in the nonempty graph J would force a guard into J. The projections therefore form an eternal family in J. □

Lemma 2.5 (component additivity)**.** *If $G_1, \dots, G_t$ are the nonempty components of G, then*

$$\gamma(G) = \sum_j \gamma(G_j), \qquad \gamma^\infty(G) = \sum_j \gamma^\infty(G_j), \qquad \theta(G) = \sum_j \theta(G_j). \tag{2.5}$$

Proof. The assertions for γ and θ follow by restriction and union. Products of componentwise eternal families prove the upper bound for γ^∞.

For the reverse bound, fix a guard-count vector $(|D \cap V(G_1)|, \dots, |D \cap V(G_t)|)$ occurring in an eternal family and retain all states with that vector. A one-edge move cannot cross components, so this subfamily remains closed. Projection to each component is an eternal family with the corresponding number of guards. Summing the componentwise lower bounds proves equality. □

Corollary 2.6 (connected reduction)**.** *If a counterexample to* (1.1) *exists, then a connected counterexample exists. Every minimum-order counterexample is connected.*

Proof. If $\gamma(G) = \gamma^\infty(G)$, additivity and $\gamma(G_j) \le \gamma^\infty(G_j)$ force equality on every component. If also $\gamma^\infty(G) < \theta(G)$, at least one component has the same strict gap. □

We next record why the common parameter of a counterexample is at least three. The following two small values use exactly the model of Definition 2.1.

Lemma 2.7. *For every odd $\ell \ge 5$,*

$$\gamma^\infty(\overline{C_\ell}) = 3. \tag{2.6}$$

Proof. Since $\theta(\overline{C_\ell}) = \chi(C_\ell) = 3$, the upper bound follows from (2.3). Label C_ℓ cyclically by $0, \dots, \ell - 1$. A pair fails to dominate $\overline{C_\ell}$ exactly when its vertices have cyclic distance two.

Suppose an eternal two-family existed. The independent pair $\{0, 1\}$ belongs to it by Lemma 2.2. From $\{0, d\}$, for odd $1 \le d \le \ell - 6$, attack $d + 2$. Moving the guard at 0 leaves the nondominating pair $\{d, d + 2\}$, so closure forces $\{0, d + 2\}$. Iteration reaches $\{0, \ell - 4\}$. Attacking $\ell - 2$ then leaves a cyclic-distance-two pair whichever guard moves. For $\ell = 5$, this last attack applies to the initial pair. □

Lemma 2.8. $\gamma^\infty(C_7) = 4$.

Proof. A partition into three edges and one singleton gives the upper bound four. Suppose an eternal family of dominating triples existed. A dominating triple has cyclic gap multiset $\{2, 1, 1\}$ or $\{2, 2, 0\}$. A triple of the second type is equivalent to $\{0, 1, 4\}$; attacking 3 forces the guard at 4 to move, after which vertex 5 is undominated. Hence no second-type triple belongs to the family.

The maximum independent set $\{0, 2, 4\}$ belongs to the family by Lemma 2.2. Attack 1. Moving the guard at 0 leaves vertex 6 undominated, while moving the guard at 2 produces the forbidden triple $\{0, 1, 4\}$, a contradiction. □

Proposition 2.9 (minimum parameter)**.** *Every counterexample satisfies*

$$\gamma(G) = \alpha(G) = \gamma^\infty(G) \ge 3. \tag{2.7}$$

Proof. The equalities follow from Proposition 2.3. Common value one forces $\alpha = 1$, so G is complete and $\theta = 1$.

Suppose the common value is two and put $H = \overline{G}$. Then $\omega(H) = 2 < \chi(H)$. By the Strong Perfect Graph Theorem [1], H contains an induced odd hole or odd antihole. An odd hole in H induces an odd antihole in G, contradicting Lemmas 2.4 and 2.7. An odd antihole in H induces an odd hole in G. At length at least seven this contradicts $\alpha(G) = 2$; at length five it again contradicts Lemma 2.7, since C_5 is self-complementary. □

2.1 The half-order restriction

We use the characterization, due independently to Payan–Xuong and to Fink–Jacobson–Kinch–Roberts, of graphs without isolated vertices whose domination number is half their order [9, 2]: every component is either C_4 or a corona $F \circ K_1$, where one leaf is attached to each vertex of a connected graph F.

Proposition 2.10. *If G has no isolated vertices and $\gamma(G) = |V(G)|/2$, then $\gamma(G) = \theta(G)$.*

Proof. For C_4, both parameters are two. If $Q = F \circ K_1$ and $|V(F)| = m$, every dominating set must contain a member of each support–leaf pair, while all supports form a dominating m-set. Hence $\gamma(Q) = m$. The m support–leaf edges partition Q into cliques, so $\theta(Q) \leq m$, while the m leaves are independent and give $\theta(Q) \geq m$. Thus the two parameters agree on every component in the characterization, and additivity completes the proof. ☐

Corollary 2.11. *If a connected graph G of order $n \geq 2$ is a counterexample with common parameter k, then*

$$n \geq 2k + 1. \tag{2.8}$$

Proof. A spanning tree of G is bipartite, and its smaller color class is a dominating set of the tree and hence of G. Thus $\gamma(G) \leq n/2$. Equality would contradict Proposition 2.10, so $k < n/2$. Integrality gives (2.8). ☐

Corollary 2.12. *A connected order-12 counterexample has common parameter*

$$k \in \{3, 4, 5\}. \tag{2.9}$$

Proof. Proposition 2.9 gives $k \geq 3$, and Corollary 2.11 gives $12 \geq 2k + 1$, hence $k \leq 5$. ☐

3 The simplicial closed-neighborhood reduction

A vertex v is *simplicial* if $N[v]$ is a clique. The next theorem strengthens the leaf reduction needed later. Its proof is included both for completeness and to expose where the one-guard dynamics enter.

Theorem 3.1 (simplicial reduction)**.** *Let G satisfy*

$$\gamma(G) = \gamma^{\infty}(G) = k,$$

let v be simplicial, and suppose $Q = G - N[v]$ is nonempty. Then Q is well-covered and

$$\gamma(Q) = \alpha(Q) = \gamma^{\infty}(Q) = k - 1, \tag{3.1}$$

while

$$\theta(G) = \theta(Q) + 1. \tag{3.2}$$

Proof. By Proposition 2.3, G is well-covered with $\gamma(G) = \alpha(G) = k$. If I is a maximal independent set of Q, then $I \cup \{v\}$ is a maximal independent set of G: the vertex v blocks $N(v)$, and I blocks $Q - I$. Hence every maximal independent set of Q has size $k - 1$, and

$$i(Q) = \alpha(Q) = k - 1. \tag{3.3}$$

In particular, $\gamma(Q) \le k-1$. If a set A of size at most $k-2$ dominated Q, then $A \cup \{v\}$ would dominate G, contradicting $\gamma(G) = k$. Therefore $\gamma(Q) = k-1$.

Fix an eternal k-family $\mathcal{D}$ in G, and choose a maximum independent set I of Q. Lemma 2.2 places $I \cup \{v\}$ in $\mathcal{D}$. Define

$$\mathcal{E} = \{D - \{v\} : D \in \mathcal{D},\ v \in D\}. \tag{3.4}$$

This family is nonempty. No state $D \in \mathcal{D}$ containing v can also contain a vertex $u \in N(v)$. If it did, $D - \{v\}$ would still dominate G: because $N[v]$ is a clique, u dominates everything that v could dominate, while v has no neighbor in Q. This would contradict $\gamma(G) = k$.

Thus each member of $\mathcal{E}$ is a $(k-1)$-subset of Q, and it dominates Q. Let $B = D - \{v\} \in \mathcal{E}$ and attack $r \in V(Q) - B$. In $D = B \cup \{v\}$, the responding guard cannot be v, since v has no neighbor in Q. The response therefore keeps v occupied and projects to

$$(B - \{u\}) \cup \{r\} \in \mathcal{E}.$$

Consequently $\mathcal{E}$ is an eternal $(k-1)$-family in Q. Together with (3.3) and $\alpha \le \gamma^\infty$, this proves (3.1).

The clique $N[v]$, together with a minimum clique partition of Q, gives $\theta(G) \le \theta(Q) + 1$. Conversely, start with a minimum clique partition of G. Replace all parts meeting $N[v]$ by the single clique $N[v]$ and the nonempty remainders after deleting $N[v]$. The old part containing v has empty remainder, so the replacement does not increase the number of parts. By minimality it is again a minimum partition. Deleting its $N[v]$ part leaves a partition of Q into $\theta(G) - 1$ cliques. Hence $\theta(Q) \le \theta(G) - 1$, proving (3.2). □

Corollary 3.2 (counterexample preservation)**.** *Under the hypotheses of Theorem 3.1, if*

$$\gamma(G) = \gamma^\infty(G) = k < \theta(G),$$

then $Q = G - N[v]$ is a smaller counterexample with common parameter $k-1$.

Proof. Theorem 3.1 gives $\gamma(Q) = \gamma^\infty(Q) = k-1$ and $\theta(Q) = \theta(G) - 1 \ge k$. □

Corollary 3.3 (leaf–support reduction)**.** *Let x be a leaf with unique neighbor y, and suppose $G - \{x, y\}$ is nonempty. If $\gamma(G) = \gamma^\infty(G) = k$, then $G - \{x, y\}$ is well-covered,*

$$\gamma(G - \{x, y\}) = \alpha(G - \{x, y\}) = \gamma^\infty(G - \{x, y\}) = k - 1,$$

and

$$\theta(G) = \theta(G - \{x, y\}) + 1.$$

A strict γ–θ gap is preserved.

Proof. The leaf x is simplicial and $N[x] = \{x, y\}$, so this is Theorem 3.1 and Corollary 3.2. □

Corollary 3.4. *Every minimum-order counterexample is connected, has no simplicial vertex, and has minimum degree at least two. Moreover, the two neighbors of every degree-two vertex are nonadjacent.*

Proof. Connectedness is Corollary 2.6. A simplicial vertex would give a smaller counterexample by Corollary 3.2, unless $G - N[v]$ were empty; in that case $G = N[v]$ would be complete. Hence there is no simplicial vertex. In a connected nontrivial graph, a leaf is simplicial, so the minimum degree is at least two. A degree-two vertex whose neighbors are adjacent is also simplicial. □

McCuaig and Shepherd proved that every connected n-vertex graph R with $\delta(R) \geq 2$, except for seven graphs of orders four and seven, satisfies

$$\gamma(R) \leq \frac{2n}{5} \tag{3.5}$$

[8]. The exception inventory is explicitly restated as $\mathcal{F}_4 \cup \mathcal{F}_7$ by Henning, Schiermeyer, and Yeo [3, Theorem 1].

Theorem 3.5 (minimum-order bound)**.** *Assume the published absence of counterexamples through order* 11*. If G is a minimum-order counterexample of order n, with*

$$k = \gamma(G) = \gamma^{\infty}(G),$$

then

$$n \geq \left\lceil \frac{5k}{2} \right\rceil. \tag{3.6}$$

Proof. The published premise gives $n \geq 12$, so G is not one of the seven exceptions of orders four and seven. Corollary 3.4 gives connectedness and minimum degree at least two. Apply (3.5) and rearrange:

$$k = \gamma(G) \leq \frac{2n}{5}.$$

Integrality gives (3.6). □

Corollary 3.6 (parameter five at order 12)**.** *Assume the published absence of counterexamples through order* 11*. There is no order-12 graph satisfying*

$$\gamma(G) = \gamma^{\infty}(G) = 5 < \theta(G).$$

Proof. Such a graph would have minimum possible order, so Theorem 3.5 would give

$$12 \geq \left\lceil \frac{25}{2} \right\rceil = 13,$$

a contradiction. □

3.1 A general independent-antineighborhood projection

The simplicial hypothesis is needed for the clique-cover identity (3.2), but not for projecting the three equality parameters or the one-guard strategy. For $A \subseteq V(G)$, write

$$N[A] = \bigcup_{a \in A} N[a].$$

Theorem 3.7 (independent-antineighborhood projection)**.** *Let $\gamma(G) = \gamma^{\infty}(G) = k$, let A be an independent set of size $t < k$, and put $Q = G - N[A]$. Then Q is nonempty and well-covered, and*

$$\gamma(Q) = \alpha(Q) = \gamma^{\infty}(Q) = k - t. \tag{3.7}$$

Moreover, every eternal k-family on G has an explicit restricted slice that is an eternal $(k-t)$-family on Q.

Proof. By Proposition 2.3, every maximal independent set of G has size k. If I is any maximal independent set of Q, then $I \cup A$ is maximal in G: A blocks $N[A] - A$, and I blocks $Q - I$. Hence every maximal independent set of Q has size $k - t$. Thus Q is nonempty and well-covered, with

$$i(Q) = \alpha(Q) = k - t. \tag{3.8}$$

If B dominated Q with at most $k-t-1$ vertices, then $A \cup B$ would dominate G with at most $k-1$ vertices. Consequently $\gamma(Q) = k - t$.

Fix an arbitrary eternal k-family $\mathcal{D}$ on G. Choose a maximum independent set I of Q. Lemma 2.2 gives $A \cup I \in \mathcal{D}$, so

$$\mathcal{E} = \{D - A : D \in \mathcal{D},\ A \subseteq D,\ D - A \subseteq V(Q)\} \tag{3.9}$$

is nonempty. If $C \in \mathcal{E}$, then $A \cup C$ dominates G. No member of A has a neighbor in Q, so C dominates Q.

For an arbitrary unoccupied attack $r \in V(Q) - C$, closure of $\mathcal{D}$ supplies a guard $u \in (A \cup C) \cap N(r)$ whose one-edge move to r stays in $\mathcal{D}$. Again no member of A is adjacent to r, so $u \in C$; the successor retains A and all other guards remain in Q. It therefore projects to

$$(C - \{u\}) \cup \{r\} \in \mathcal{E}.$$

This proves the exact $\forall C\, \forall r\, \exists u$ closure condition on Q, using one guard and one edge. Hence $\gamma^\infty(Q) \leq k - t$, while $\alpha(Q) \leq \gamma^\infty(Q)$ and (3.8) give equality. □

Corollary 3.8. *If G is a minimum-order counterexample with common parameter k, then for every nonempty independent t-set A, $t < k$,*

$$\gamma(G - N[A]) = \alpha(G - N[A]) = \gamma^\infty(G - N[A]) = \theta(G - N[A]) = k - t. \tag{3.10}$$

Equivalently, if $H = \overline{G}$, then for every t-clique A of H,

$$\chi(H[N_H(A)]) = \omega(H[N_H(A)]) = k - t, \tag{3.11}$$

where $N_H(A) = \bigcap_{a \in A} N_H(a)$.

Proof. Theorem 3.7 gives every equality in (3.10) except the one involving θ. If $\theta(G - N[A]) > k - t$, the proper induced graph $G - N[A]$ would be a smaller counterexample. Minimality gives equality. Complementation translates $G - N[A]$ to $H[N_H(A)]$, proving (3.11). □

Remark 3.9 (prior-art and strength boundary). Taletskii's Lemma 13 already proves essentially (3.10) for a minimum-order planar counterexample [10]. Theorem 3.7 removes planarity and the hypothesis that the graph is a minimum counterexample from the $\gamma, \alpha, \gamma^\infty$ projection and explicitly projects every eternal family. It is therefore presented as an unrestricted, family-level generalization, not as a wholly new antineighborhood idea. Its full independent set form is also obtained by iterating the $t = 1$ case. For $k = 3$, (3.11) is exactly local bipartiteness together with the well-covered clique structure, so it gives no new pruning beyond the odd-wheel obstruction already used in the parameter three analysis. Novelty relative to the unavailable 2018 manuscript remains unresolved.

4 The certified order-12 parameter-three slice

Theorem 4.1. *There is no finite simple graph G of order 12 satisfying*

$$\gamma(G) = \gamma^{\infty}(G) = 3 < \theta(G). \tag{4.1}$$

This assertion includes disconnected graphs.

We summarize the complete coverage proof and exact certificates. Full proof details and all bindings are in the accompanying archive.

4.1 The exhaustive complement split

Call a vertex outside an induced cycle a *hub* if it is adjacent to every rim vertex. A cycle is *hub-free* if it has no hub.

Proposition 4.2 (odd-wheel obstruction)**.** *If $\gamma^{\infty}(G) = 3$, then $\overline{G}$ contains no induced odd wheel.*

Proof. If $\overline{G}$ induces $K_1 \vee C_{2q+1}$, then G induces $K_1 \dot\cup \overline{C_{2q+1}}$. Lemmas 2.5 and 2.7 give eternal domination number four for this induced subgraph, contradicting Lemma 2.4. □

Proposition 4.3 (three-template split)**.** *If G satisfies (4.1), then G is connected and $H = \overline{G}$ contains a hub-free induced C_ℓ for some*

$$\ell \in \{5, 7, 9\}. \tag{4.2}$$

Proof. If G were disconnected, component additivity would give a counterexample component. Proposition 2.9 says that component has parameter at least three, so it consumes the entire domination budget; hence G is connected.

Proposition 2.3 gives

$$\omega(H) = \alpha(G) = 3 < \chi(H) = \theta(G). \tag{4.3}$$

The Strong Perfect Graph Theorem supplies an induced odd hole or odd antihole. An odd antihole of length $2q + 1$ has clique number q, so its length is five or seven. The five-antihole is C_5. An induced $\overline{C_7}$ in H induces C_7 in G, contradicting Lemmas 2.4 and 2.8. Thus H contains an odd hole.

It is hub-free by Proposition 4.2. Since no pair dominates G, every pair in H has a common H-neighbor. The endpoints of a rim edge have no common neighbor on an induced rim, so they have one outside it. If only one vertex lay outside the hole, it would be a hub. At least two vertices therefore lie outside, and the odd rim has length at most ten. This gives (4.2). □

4.2 Exact formulas and coverage

Fix $\ell \in \{5, 7, 9\}$, label an induced cycle of $H = \overline{G}$ by $0, \ldots, \ell - 1$, and label one common H-neighbor of rim edge 01 as vertex ℓ. The formula has 6,886 variables:

$$\begin{array}{rcll} 66 & : & e_{uv} & (uv \in E(H)), \\ 660 & : & w_{ab,c} & (c \text{ witnesses a common } H\text{-neighbor of } a, b), \\ 220 & : & f_D & (D \in \binom{[12]}{3} \text{ is selected}), \\ 5,940 & : & m_{D,r,u} & (u \in D \text{ responds to } r \notin D). \end{array} \tag{4.4}$$

The clauses assert: no K_4 in H; a common H-neighbor for every pair; the labeled hub-free cycle template; connectedness of G; a nonempty selected family of dominating triples; one legal successor for every selected state and every unoccupied attack; and selection of every H-triangle. In particular, a move literal implies $\neg e_{ur}$, because a legal move uses an edge of G, not H. It also selects the state $(D - \{u\}) \cup \{r\}$, whose own clauses require domination.

For every normalized map $c : [12] \to \{0, 1, 2\}$ proper on the forced template, the coloring obstruction is

$$C_c = \bigvee_{\substack{u<v \\ c(u)=c(v)}} e_{uv}. \tag{4.5}$$

It is false exactly when c is a proper coloring of H. The C_5 and C_7 formulas contain complete banks of respectively 3,645 and 1,701 such clauses. The C_9 formula contains 170 valid coloring clauses obtained during counterexample-guided refinement. No completeness claim is needed for that subset: every graph with $\chi(H) > 3$ satisfies every clause (4.5).

The C_5 branch also lexicographically orders the six undistinguished outer signatures. The formula is invariant under their full permutation group, so every model has an orbit representative satisfying the order.

Lemma 4.4 (parameter-three graph-to-formula coverage)**.** *Every graph satisfying (4.1) induces a satisfying assignment of at least one of the three exact formulas in Table 1.*

Proof. Use Proposition 4.3 and label the resulting template as above. Set edge and witness variables from H. Equation (4.3) supplies the no-K_4 and coloring clauses; failure of every pair to dominate supplies the common-neighbor clauses; and connectedness and hub-freeness supply the corresponding template clauses.

Set f_D on an actual nonempty eternal three-family and choose one one-guard response for each selected state and unoccupied attack. Lemma 2.2 selects every H-triangle. If $\ell = 5$, permute undistinguished vertices and all associated semantic variables to satisfy the signature order. Thus the exact branch formula is satisfied. ◻

4.3 Certificates

Table 1: Exact parameter-three branch formulas and accepted addition-only RUP proofs.

	C_5	C_7	C_9
variables	6,886	6,886	6,886
clauses	23,968	21,718	20,200
literals	192,169	148,551	117,841
CNF bytes	754,323	621,864	530,053
proof additions	247,981	284,317	4,705
proof bytes	6,337,621	18,093,724	65,906

In C_5, C_7, C_9 order, the formula SHA-256 values are

```
c6a0811c718ff8e9352f253e4ce225ce2826def9c3e4a9cd55f0b0152703d104
6a011e685e58ef517f2ab8253ca40987bd7b742a470bedbacdc3a5e94fc995a7
2845f242a094484a8d114e70ca1a8678dfcff79fadd56bd57813e25c2e49523d
```

The corresponding addition-only proof SHA-256 values are

```
c6c24853e30073e66fb396441edb176a0160d062a8558e25fa18a955f33927c3
e8052df40d3e0c39b945a8735889039daba55eacc351e1822828b3d94f7baae9
24c5647d3a57f2de221fba96747c618575a3aba086c5e4bca17aade55ce7d4ab
```

Each proof is an addition-only sequence of reverse-unit-propagation (RUP) consequences ending in the empty clause, so it is a directly checkable unsatisfiability certificate [11]. Strict replays reported zero RAT lemmas. The C_5 binary proof was independently parsed and replayed; the C_7 proof passed two warning-fatal RUP-only replays; and the C_9 proof passed both a separately written standard-library checker and a pinned external checker. Two independent reviews audited the three-branch coverage theorem.

Proof of Theorem 4.1. A putative graph gives a satisfying assignment of one exact formula by Lemma 4.4. The corresponding accepted RUP refutation proves that formula unsatisfiable, a contradiction. □

5 The certified connected order-12 parameter-four slice

Theorem 5.1. *There is no connected finite simple graph G of order 12 satisfying*

$$\gamma(G) = \gamma^{\infty}(G) = 4 < \theta(G). \tag{5.1}$$

Put $H = \overline{G}$, and let $e_{uv} = 1$ mean $uv \in E(H)$. Equality collapse gives

$$\gamma(G) = \alpha(G) = \gamma^{\infty}(G) = 4 < \theta(G). \tag{5.2}$$

5.1 An exact anchored synthesis formula

Proposition 5.2 (static characterization)**.** *Up to relabeling vertices, $\gamma(G) = \alpha(G) = 4$ if and only if:*

1. *$A = \{0, 1, 2, 3\}$ is a K_4 in H;*
2. *H has no K_5; and*
3. *every three-set S has a common H-neighbor outside S.*

Proof. If $\gamma = \alpha = 4$, label a maximum independent four-set of G as A. It is a K_4 of H, and $\alpha(G) = 4$ forbids a K_5 in H. No three-set dominates G, so every three-set has a vertex adjacent in G to none of it, equivalently a common H-neighbor.

Conversely, the anchor and no-K_5 clauses give $\alpha(G) = 4$. The common-neighbor condition says that no three-set dominates G, so $\gamma(G) \geq 4$. The independent four-set A is maximum and hence maximal, so it dominates G and $\gamma(G) \leq 4$. □

The exact formula uses the following 18,381 variables.

role	family	count
complement edges	$e_{uv},\ u < v$	66
triple common-neighbor witnesses	$w_{S,x},\ \lvert S\rvert = 3,\ x \notin S$	1,980
selected guard states	$f_D,\ \lvert D\rvert = 4$	495
one-guard responses	$m_{D,r,u},\ r \notin D,\ u \in D$	15,840
total		18,381

(5.3)

Its static clauses encode Proposition 5.2 and connectedness of G. For every nonempty proper cut S containing vertex 0, the connectedness clause is

$$\bigvee_{\substack{u\in S\\ v\notin S}} \neg e_{uv}, \tag{5.4}$$

where a negative H-edge is a G-edge.

For each four-set D and $x \notin D$, domination is

$$\neg f_D \ \vee \ \bigvee_{u\in D} \neg e_{ux}. \tag{5.5}$$

At least one f_D is true. For each selected D and unoccupied attack $r \notin D$, the clauses require at least one $m_{D,r,u}$, and each true response implies

$$\neg e_{ur} \quad \text{and} \quad f_{(D-\{u\})\cup\{r\}}. \tag{5.6}$$

Thus the encoded quantifier is exactly

$$\forall D \in \mathcal{D}\ \forall r \notin D\ \exists u \in D,$$

with one named guard moving along one G-edge to a selected successor. Every H-K_4 is also selected, a sound strengthening by Lemma 2.2.

The anchored K_4 fixes four distinct colors in any proper four-coloring of H. Normalize them by $c(j) = j$ for $j \in A$. For each of the $4^8 = 65,536$ colorings of the remaining vertices, append

$$C_c = \bigvee_{\substack{u<v\\ c(u)=c(v)}} e_{uv}. \tag{5.7}$$

The full bank is equivalent to $\chi(H) > 4$, hence to $\theta(G) > 4$.

Finally, lexicographically order the four-bit anchor signatures

$$(e_{0v}, e_{1v}, e_{2v}, e_{3v}), \qquad v = 4, \ldots, 11. \tag{5.8}$$

Permuting the eight outer vertices and all semantic variables preserves the base formula and permutes the complete coloring bank, so every orbit has an ordered representative.

Proposition 5.3 (parent coverage)**.** *The resulting anchored parent formula is satisfiable if and only if, up to relabeling one maximum independent four-set, there is a connected 12-vertex graph satisfying (5.2).*

Proof. Given such a graph, Proposition 5.2 supplies the static assignment, an actual eternal family supplies (5.5)–(5.6), and $\chi(H) > 4$ supplies (5.7). Relabel outer vertices to satisfy (5.8).

Conversely, the static clauses give $\gamma = \alpha = 4$, the selected-state clauses give an eternal four-family and hence $\gamma^\infty = 4$, the cut clauses give connectedness, and the complete bank gives $\theta > 4$. $\square$

The exact parent DIMACS file has

$$\begin{aligned} \text{variables} &= 18,381, \\ \text{clauses} &= 114,742, \\ \text{literal occurrences} &= 1,180,016, \\ \text{bytes} &= 3,992,947. \end{aligned} \tag{5.9}$$

Its SHA-256 is

`adbe0c01614bae6cd3aed4ccdcd45a757ca56e7ef9c4f2f280f2d8ef200e40ac`.

5.2 DoubleLex orbit reduction

Form the 8-by-4 anchor–outer incidence matrix

$$M_{v,j} = e_{jv}, \qquad v \in \{4, \dots, 11\},\ j \in A. \tag{5.10}$$

The group $S_8 \times S_4$ permutes its rows and columns. Relabeling vertices simultaneously in all graph, witness, family, and move variables preserves the formula. Under an anchor permutation, permuting the four color names restores the normalized anchor colors, so the complete bank is also invariant.

Proposition 5.4 (DoubleLex equisatisfiability)**.** *The parent formula is satisfiable if and only if it has a model in which both the rows and columns of M are lexicographically nondecreasing.*

Proof. Choose, among the finite $S_8 \times S_4$ orbit of a parent model, an image whose row-major matrix word is lexicographically least. If an earlier row exceeded a later row, swapping those rows would decrease the word. Hence the rows are ordered. If adjacent columns were inverted, swap them. At their first differing row, the row-major word changes from 1 to 0, with all earlier entries fixed, again a contradiction. Thus the same least image has both orders. The reverse implication is immediate. □

Three auxiliary-free eight-bit comparators impose the column order. They add 765 clauses and 10,758 literals. The exact strengthened formula D therefore has

$$\begin{aligned} \text{variables} &= 18,381, \\ \text{clauses} &= 115,507, \\ \text{literal occurrences} &= 1,190,774, \\ \text{bytes} &= 4,030,657. \end{aligned} \tag{5.11}$$

The exact SHA-256 binding is

`14284db1f0b9cfb37b91d834fbabac1d0ca06d36e0d2782683e35cbd04a976e7`.

An independent reconstruction reproduced both the complete parent prefix and the 37,710-byte comparator suffix byte for byte. Exhaustive comparator tests and semantic mutation tests checked the symmetry and one-guard encoding.

5.3 The exact LRAT refutation

A complete solver proof was normalized to an addition-only RUP stream of 15,783,377 bytes with SHA-256

`2741335a5ed9af769f0db4bd0c03a70e414d0568681d5b8261a5667ed30b6686`.

Warning-fatal forward and backward replays both verified the stream and reported zero RAT lemmas. Conversion produced a 228,381,671-byte LRAT certificate with SHA-256

`0e04eb639a3f7f7d335126d56040abb4ef11e8548770262c316a608390659263`.

The pinned, separately checked `lrat-check` executable accepted it.

The independent hostile audit reconstructed D and the normalized proof byte for byte, reran both RUP directions, generated a fresh byte-identical LRAT, replayed both retained and fresh LRATs, and rejected six mutated artifacts. Its exact verdict was

`ACCEPT_EXACT_DOUBLELEX_CNF_UNSAT_ONLY`.

A separate implication audit checked the transfer from exact D-UNSAT through Propositions 5.3 and 5.4.

Proof of Theorem 5.1. If a connected graph satisfied (5.1), equality collapse would give (5.2), and Proposition 5.3 would give a parent model. Proposition 5.4 would then give a model of D. The accepted LRAT certificate proves D unsatisfiable, a contradiction. □

Remark 5.5 (connected scope). Theorem 5.1 is a connected exclusion. It does not, by itself, exclude every disconnected parameter-four graph of order 12. In the proof of Theorem 1.1, the published absence of smaller counterexamples makes a putative order-12 counterexample minimum-order, and Corollary 2.6 then supplies connectedness.

6 Assembly of the order-12 frontier

Proof of Theorem 1.1. Orders at most 11 are excluded by the explicitly assumed published computation. Suppose an order-12 counterexample G exists. It then has minimum possible order and is connected by Corollary 2.6. Put

$$k = \gamma(G) = \gamma^{\infty}(G).$$

Corollary 2.12 leaves exactly

$$k \in \{3, 4, 5\}.$$

Theorem 4.1 excludes $k = 3$; Theorem 5.1 excludes the connected $k = 4$ case; and Corollary 3.6 excludes $k = 5$. No parameter remains. □

Remark 6.1 (evidentiary boundary). The $k = 3$ and connected $k = 4$ order-12 slices are backed by campaign certificates and independent replays. The $k = 5$ slice is analytic, relative to the classical McCuaig–Shepherd theorem and the published through-order-11 computation. The all-parameter statement through order 12 additionally imports that published computation at orders at most 11. The campaign's audit of the published 56-graph appendix is a regression check, not a replacement for the original all-graph coverage.

7 Certificates, reproduction, and provenance

All paths below are relative to the archive root `gamma_theta_eternal_domination/`.

7.1 Parameter-three replay

The accepted theorem note and its two independent composition reviews are:

```
math/lemmas/order12_k3_exclusion.md
reviews/order12_k3_exclusion_hostile_review.md
reviews/order12_k3_exclusion_second_review.md.
```

The claim-level acceptance record is

`results/order12_k3_exclusion_acceptance.json`.

From the archive root, the fail-closed full replay is:

```
python3 -I -B repro/c035/replay.py --mode full \
  --output /fresh/path/c035-full-replay.json
```

It validates exact source, formula, proof, checker, review, and manifest bindings before running the three proof replays in fresh private directories. A successful record says `PASS_FULL_C035_REPLAY` and `proofs_freshly_replayed=true`. The fast mode checks metadata only and explicitly returns no mathematical claim.

7.2 Parameter-four replay

The compact publication package is

`certificates/order12_k4_doublelex_seed0_lrat_publication/`.

It contains the 64,288,636-byte compressed LRAT with SHA-256

`edc0f6b76bc96b2b26677f399566ae437cc47a6fc0cc921eaff81d77b72a50da`.

Decompression is accepted only if the result has the size and hash in Section 5. From the archive root, the one-command replay is:

```
python3 \
  certificates/order12_k4_doublelex_seed0_lrat_publication/\
verify_publication.py
```

The verifier hashes the formula, package metadata, compressed and recovered proofs, checker, author package, and independent review before invoking `lrat-check`. Its accepted terminal verdict is `VERIFIED_EXACT_DOUBLELEX_CNF_UNSAT_ONLY`. The graph transfer is separately recorded in:

```
  math/lemmas/order12_k4_synthesis_target.md
  math/lemmas/order12_k4_doublelex.md
  reviews/order12_k4_doublelex_conditional_implication_audit/
    REVIEW.md.
```

7.3 Frontier and structural reviews

The assembled theorem and the hash-bound independent review are:

```
  math/lemmas/order12_frontier.md
  reviews/order12_frontier_second_review/REVIEW.md.
```

Their SHA-256 values are respectively

`adb27204d33feb47933f2a4b1e381485b2e1b80c22b56a67b18586c4933c2b75`

and

`875447d219e5670d66be7d6e4b7ec9f9b3b03bba4d2b169b836706b624a92926`.

The review's exact verdict is `ACCEPT_ORDER12_FRONTIER_WITH_EXPLICIT_PUBLISHED_PREMISE`. The simplicial theorem was separately audited in

`reviews/simplicial_neighborhood_reduction_hostile/`.

That review found no mathematical defect and exhaustively tested the reduction on all 13,598 graphs in the retained small-graph universe through order 8. Finite testing is supporting regression evidence; the proof of Theorem 3.1 is the mathematical evidence.

The independent-antineighborhood theorem, its exact prior-art audit, and its clean-room falsification probe are:

```
math/lemmas/independent_antineighborhood_projection.md
reviews/general_neighborhood_projection_independent/REVIEW.md
reviews/general_neighborhood_projection_independent/probe.py
```
.

The probe imports no campaign evaluator and checks all 13,598 unlabeled graphs through order 8, including 14,421 eligible independent sets and 56,166 one-guard attack obligations. Its role is falsification and regression; the proof of Theorem 3.7 is the unbounded-order evidence.

7.4 Research and verification provenance

This manuscript and the associated research artifacts were produced with substantial assistance from autonomous AI systems. No theorem or finite claim was accepted on model output alone. Mathematical reductions were assigned to separate adversarial proof reviews; exact formulas were independently reconstructed; proof streams were replayed by checkers that do not import the search implementation's graph-transition core; and mutation tests were used to detect complement-sign, occupied-attack, all-guards-move, missing-successor, and malformed-certificate errors. These procedures reduce, but do not eliminate, the need for human inspection. Human authors remain responsible for any submitted version and for the archived evidence.

Data availability

The formulas, proofs, verifiers, manifests, reviews, literature-source identifiers, and exact hashes cited above are publicly archived in the tagged release `gamma-theta-order12-frontier-v1.0.0`. The corresponding human-readable project page records the active campaign status.

Acknowledgment of limitations

No external expert was contacted during this independent campaign. The universal conjecture remains open in this work. The strongest conclusion is the explicitly delimited order-12 frontier theorem, and its lower-order part relies on the published computation of MacGillivray–Mynhardt–Virgile.

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