Semiregular C_{37} Conference Lifts at Order 334: Quotient Classification, Modular Realizations, and Low-Rank Obstructions

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Abstract

We study symmetric conference matrices of order 334 whose 333-vertex core admits a semiregular cyclic group of order 37, with nine vertex orbits. Such a matrix would yield a Hadamard matrix of order 668, but no such matrix is constructed here.

The zero-frequency orbit equations are classified exhaustively. There are 196,560,000 labeled orbit-sum matrices, 625 classes under simultaneous permutation, and 314 classes after also identifying global sign. Their reductions modulo two collapse to three parity classes, or two up to complement. No quotient has zero diagonal, and a Fourier trace argument therefore forces a universal 6/3 incidence law on the diagonal circulant blocks.

We count exact finite-field relaxations. In characteristic two the nontrivial Fourier component is a rank-four Hermitian projection in a nine-dimensional unitary space over $\mathbb{F}_{2^{36}}/\mathbb{F}_{2^{18}}$; the orientation-fixed relaxation has between 2^{719} and 2^{720} points. In characteristic three the corresponding Hermitian square-zero census lies between 2^{1141} and 2^{1142} per fixed quotient. We also construct an exact-margin binary support for one integral representative of each of the two complement classes. Both supports satisfy the complete graph equation modulo two, equivalently the conference-core equation modulo eight.

In characteristic 37, the first-moment system has rank 16, and for a certified quotient every admissible first moment has an explicit formal completion through all 37 truncated layers. Physical 0/1-realizability remains the obstruction. Within the resulting exponential family, all constant symmetric generators of ranks at most three are excluded across all 625 quotients. At the first nonconstant layer we obtain a $20 + 20$-parameter normal form and exclude both pure rank-two higher terms and the fully enumerated common nondegenerate two-plane pencil. Finally, exhaustive four-cycle and small semiregular-transvection gates fail to repair either frozen characteristic-two support.

The retained supports do not satisfy the adjacency equation modulo four: the best recorded carry has 672 nonzero coordinates among 1503. Consequently this paper neither constructs nor excludes a conference graph, and it does not settle the Hadamard problem at order 668.

1 Introduction

A normalized symmetric conference matrix of order 334 may be written

$$C = \begin{pmatrix} 0 & \mathbf{1}^T \\ \mathbf{1} & S \end{pmatrix}, \qquad C^2 = 333 I_{334}.$$

Its symmetric 333×333 core satisfies

$$S\mathbf{1} = 0, \qquad S^2 = 333I_{333} - J_{333}. \tag{1.1}$$

Equivalently,

$$A = \frac{J - I - S}{2}$$

would be a strongly regular graph with parameters

$$(v, k, \lambda, \mu) = (333, 166, 82, 83). \tag{1.2}$$

The standard conference doubling construction would then produce a Hadamard matrix of order 668; the conference-matrix correspondence and doubling construction are classical [5].

At the date of this draft, order 668 remains listed as the smallest unresolved Hadamard order in both a recent construction database and a public open-problem list [3, 4], and the strongly regular graph parameter set (1.2) remains open in the standard table [2]. Mathon's title "Symmetric conference matrices of order $pq^2 + 1$" is a potentially misleading near-match because $333 = 37 \cdot 3^2$. In his Theorem 4.1, however, the same integer t determines $q = 4t - 1$ and $p = 4t + 1$, so $p = q + 2$; the pair $(p, q) = (37, 3)$ is outside that construction [7].

The orbit-matrix method for strongly regular graphs with prime-order automorphisms is classical [1], as is the general multicirculant or m-Cayley framework [6]. A recent multiplier-obstruction theorem for Legendre pairs of length 333 concerns a different, common-multiplier route and explicitly leaves unrestricted existence open [9]. Our claims concern the stated nine-orbit calculation and its quotient-specific consequences, not those general frameworks.

We impose a semiregular C_{37}-action with nine vertex orbits. This is strictly broader than a regular action by a group of order 333. The result is a finite but still enormous 45-block lifting problem. The main contribution is to identify exactly what is already forced, what can be realized modulo small primes, and which natural formal and switching families are impossible.

Main results

The principal results are as follows.

(i) The integral zero-frequency quotient system has exactly 625 permutation classes and 314 sign-permutation classes (Theorem 3.1).

(ii) The quotient census has only three parity types, two up to complement, and every integral lift satisfies a 6/3 quadratic-residue incidence law on its diagonal blocks (Theorems 3.2 and 4.1).

(iii) The orientation-fixed characteristic-two relaxation is a unitary Grassmannian of size strictly between 2^{719} and 2^{720}. The characteristic-three relaxation is counted separately (Theorems 5.1 and 5.2).

(iv) Both complement classes have explicit exact-margin binary supports satisfying every equation modulo two (Theorem 6.1).

(v) The characteristic-37 first-moment map has rank 16. For one certified quotient every admissible first moment extends through all truncated layers to a formal solution, but not necessarily to binary blocks (Theorems 7.1 and 7.2).

(vi) Constant generators of ranks one, two, and three, as well as the diagonal constant family, are impossible in the explicit formal conjugation family. Two fully enumerated rank-two subfamilies at the first nonconstant layer are also impossible (Theorems 8.1 and 9.2).

(vii) Neither ordinary four-cycle switching nor the smallest semiregular unitary-transvection family repairs the two frozen modular supports (Theorem 10.1).

Computational assertions are tied to deterministic replay programs or finite certificates, with separate checks for several key counts; Section 11 gives a map from claims to artifacts.

Scope

The distinction between a relaxation and a construction is essential. The two binary supports satisfy the adjacency equation only modulo two. None satisfies it modulo four, and no integer conference graph, symmetric conference matrix, or Hadamard matrix is produced. The formal characteristic-37 completion similarly solves a matrix equation in a truncated group algebra while deliberately forgetting the physical 0/1 coefficient alphabet. All results are restricted to the semiregular C_{37} lane; they do not exclude that lane, much less conference graphs without such an automorphism.

2 The semiregular group-ring model

Let

$$U(x) = 1 + x + \cdots + x^{36}$$

in the group ring $\mathbb{Z}[C_{37}] \cong \mathbb{Z}[x]/(x^{37} - 1)$. A graph invariant under a semiregular C_{37}-action on nine fibers is represented by a 9×9 matrix

$$D(x) = (d_{ij}(x))$$

of binary cyclic words. Undirectedness and looplessness give

$$d_{ji}(x) = d_{ij}(x^{-1}), \qquad [x^0]d_{ii}(x) = 0. \tag{2.1}$$

The strongly regular equation is

$$D(x)^2 + D(x) = 83I_9 + 83U(x)J_9. \tag{2.2}$$

The augmentation

$$B = D(1)$$

is the equitable adjacency quotient. In Seidel coordinates define

$$T = 37J_9 - I_9 - 2B. \tag{2.3}$$

Then T is the matrix of block row sums of the core S. Evaluation of (1.1) at the trivial character gives

$$T\mathbf{1} = 0, \qquad T^2 = 333I_9 - 37J_9. \tag{2.4}$$

The diagonal entries of T are even sums of 36 signs; the off-diagonal entries are odd sums of 37 signs.

Lemma 2.1 (Local quotient rigidity)**.** *Every solution of* (2.4) *satisfying the block bounds has*

$$T_{ii} \in \{-16, -12, -8, -4, 0, 4, 8, 12, 16\}, \qquad |T_{ij}| \le 15 \quad (i \neq j).$$

Proof. For one row, write $d = T_{ii}$ and $x_1, \ldots, x_8$ for its odd off-diagonal entries. The row equations are

$$d + \sum_j x_j = 0, \qquad d^2 + \sum_j x_j^2 = 296.$$

Every odd square is 1 mod 8, so $d^2 \equiv 0 \bmod 8$, whence $4 \mid d$. The norm equation gives $|d| \le 16$. If some $|x_j| = 17$, the remaining seven odd squares and d^2 must contribute only 7, forcing $d = 0$ and the seven remaining entries to be ± 1; they cannot cancel ± 17. □

There are exactly 411 unordered row multisets satisfying these two row equations, expanding to 2,109,524 ordered rows. This finite row catalog is the input to the quotient census.

3 Complete quotient classification

Theorem 3.1 (Exact quotient census)**.** *The system* (2.4)*, together with symmetry, parity, and the block bounds, has exactly*

196,560,000	*fully labeled solutions,*
625	*classes under* $T \sim PTP^T$*,*
314	*classes under* $T \sim \pm PTP^T$*.*

Exactly three permutation classes are equivalent to their negatives. The automorphism-order distribution is

$\lvert \mathrm{Aut}(T) \rvert$	1	2	3	4	6	24
# *classes*	480	100	24	5	14	2.

Every quotient has rank 4 *over* $\mathbb{F}_{37}$*. There are* 111 *diagonal multisets, none equal to* 0^9*.*

Certified enumeration. The enumerator fills a symmetric matrix row by row from the complete ordered row catalog. A new row agrees with the previously fixed upper triangle and has inner product -37 with every earlier row. The root row is sorted. At later depths, unprocessed vertices with identical columns against earlier rows form interchangeable cells, and new entries inside each cell are required to be nondecreasing. This is an orderly augmentation in the standard isomorph-free generation sense [8]: every solution has a relabeling satisfying the imposed order.

There are 7016 orderly terminal matrices. Each terminal is checked directly against every entry of (2.4), canonicalized under all 9! permutations, and assigned a stabilizer. The labeled total is also reconstructed from the orbit-stabilizer identity

$$\sum_{[T]} \frac{9!}{|\,\mathrm{Aut}(T)|} = 196{,}560{,}000.$$

The packaged certificate `z37_quotient_census_canonical_625.txt` contains the 625 canonical upper triangles together with the audit summaries. Its SHA-256 is

`c5d8765da49deb39c2ff3407b9d0f265e3ca56c1015d5b0075355c53ca60fb5b`.

For the rank assertion, $T^2 = 0 \bmod 37$ makes $\operatorname{im} T$ totally isotropic in dimension nine. Hence $\operatorname{rank} T \le 4$. On $\mathbf{1}^\perp$, the determinant has 37-adic valuation 4. The decomposition

$$\mathbb{F}_{37}^9 = \langle \mathbf{1} \rangle \oplus \mathbf{1}^\perp$$

therefore gives $\operatorname{rank} T \ge 4$. □

An explicit feasible quotient is

$$\begin{pmatrix} 0 & -11 & -3 & 7 & -3 & 7 & -3 & 7 & -1 \\ -11 & -4 & 7 & -1 & 7 & -1 & 7 & -1 & -3 \\ -3 & 7 & 0 & -11 & -3 & 7 & -3 & 7 & -1 \\ 7 & -1 & -11 & -4 & 7 & -1 & 7 & -1 & -3 \\ -3 & 7 & -3 & 7 & 0 & -11 & -3 & 7 & -1 \\ 7 & -1 & 7 & -1 & -11 & -4 & 7 & -1 & -3 \\ -3 & 7 & -3 & 7 & -3 & 7 & 0 & -11 & -1 \\ 7 & -1 & 7 & -1 & 7 & -1 & -11 & -4 & -3 \\ -1 & -3 & -1 & -3 & -1 & -3 & -1 & -3 & 16 \end{pmatrix}. \tag{3.1}$$

It is signed-permutation equivalent to a $1+4+4$ normal form and is one of the two order-24 permutation classes. Those two classes are negatives, hence form the unique maximally symmetric sign-class.

Theorem 3.2 (Parity collapse)**.** *Reducing $B = (37J - I - T)/2$ modulo two maps the 625 integral quotient classes onto exactly three binary graph classes. Their integral preimage counts are*

$$206, \quad 213, \quad 206,$$

and their automorphism orders are

$$24, \quad 20, \quad 24.$$

The first and third are complements; the middle class is self-complementary up to permutation. Thus two types remain after identifying complement.

The number of fully labeled binary quotients is consequently

$$\frac{9!}{24} + \frac{9!}{20} + \frac{9!}{24} = 48{,}384.$$

One order-24 representative consists of a K_4, an independent set of order four, a ninth vertex joined to the independent set, and a matching between the two four-sets. Its degree sequence is $2^4, 4^5$. Its complement has degree sequence $4^5, 6^4$, while the order-20 class has degree sequence $2, 2, 4, 4, 4, 4, 4, 6, 6$.

4 The universal diagonal trace law

For $t \in \mathbb{F}_{37}^{\times}$, let $g(t)$ be the number of the nine diagonal adjacency blocks containing displacement t.

Theorem 4.1 (The 6/3 law)**.** *After possibly interchanging quadratic residues and nonresidues, every semiregular C_{37} lift of every quotient in Theorem 3.1 satisfies*

$$g(t) = \begin{cases} 6, & t \in \mathrm{QR}, \\ 3, & t \in \mathrm{NQR}. \end{cases} \tag{4.1}$$

Proof. At a nontrivial character ζ^a, the Hermitian Fourier block $D(\zeta^a)$ satisfies

$$D(\zeta^a)^2 + D(\zeta^a) = 83 I_9.$$

Its only eigenvalues are

$$r, s = \frac{-1 \pm 3\sqrt{37}}{2}.$$

At frequency zero, $B = D(1)$ has eigenvalue 166 on $\langle \mathbf{1} \rangle$. On $\mathbf{1}^{\perp}$, equation (2.4) gives the conjugate eigenvalues r, s, each with multiplicity four. The full $\mathrm{srg}(333, 166, 82, 83)$ spectrum has r and s, each with multiplicity 166.

Cyclotomic Galois conjugacy makes the multiplicity of r constant, say m, on the 18 quadratic-residue frequencies and constant, say m', on the 18 nonresidue frequencies. Counting the full r-eigenspace, including the four copies at frequency zero, gives

$$4 + 18m + 18m' = 166,$$

so $m + m' = 9$, or $m' = 9 - m$. Consequently the two nonzero-frequency traces are

$$\tau_{\mathrm{QR}} = -\frac{9}{2} + \frac{3\sqrt{37}}{2}(2m - 9), \qquad \tau_{\mathrm{NQR}} = -\frac{9}{2} - \frac{3\sqrt{37}}{2}(2m - 9),$$

while $\tau_0 = \operatorname{Tr} B = 162$.

For $t \neq 0$, the quadratic Gaussian sums, with a fixed choice of $\sqrt{37}$, are

$$\sum_{a \in \mathrm{QR}} \zeta^{-at} = \frac{-1 + \chi(t)\sqrt{37}}{2}, \qquad \sum_{a \in \mathrm{NQR}} \zeta^{-at} = \frac{-1 - \chi(t)\sqrt{37}}{2}.$$

Fourier inversion of $\operatorname{Tr} D(\zeta^a) = \sum_t g(t)\zeta^{at}$ now gives

$$g(t) = \frac{1}{37}\left(162 + \tau_{\mathrm{QR}} \sum_{a \in \mathrm{QR}} \zeta^{-at} + \tau_{\mathrm{NQR}} \sum_{a \in \mathrm{NQR}} \zeta^{-at}\right) = \frac{9 + 3(2m - 9)\chi(t)}{2}. \tag{4.2}$$

The pointwise bounds $0 \leq g(t) \leq 9$ leave $m \in \{3, 4, 5, 6\}$, and therefore, up to orientation, incidence pairs $(0, 9)$ or $(3, 6)$.

In the $(0, 9)$ branch every diagonal block is one complete quadratic class and has degree 18. Equation (2.3) would then give $T_{ii} = 0$ for all i, contradicting the exhaustive assertion in Theorem 3.1. Hence only $(3, 6)$ remains. $\square$

4.1 Ambient search sizes

There are 1494 raw block-membership bits: 1332 from the 36 unordered off-diagonal blocks and 162 inverse-pair bits from the nine diagonal blocks. For one quotient T, the fixed-margin count before (4.1) is

$$\prod_i \binom{18}{(36 - T_{ii})/4} \prod_{i<j} \binom{37}{(37 - T_{ij})/2}. \tag{4.3}$$

Across the 625 classes, the extremal base-two logarithms are

$$1340.52392808143 \quad \text{and} \quad 1340.83193919044.$$

After the 6/3 law and a nonresidue decimation fixing its orientation, the per-class logarithm lies between

$$1297.60492221007667 \quad \text{and} \quad 1297.90621474626255.$$

Summing one canonical space for each quotient gives

$$\log_2(\text{625-class trace-law union}) = 1307.10873431446430. \tag{4.4}$$

These are exact combinatorial counts before the remaining Fourier equations, not estimates of conference graphs.

5 Exact finite-field relaxations

5.1 Characteristic two

Because 2 is primitive modulo 37,

$$\mathbb{F}_2[C_{37}] \cong \mathbb{F}_2 \times K, \qquad K = \mathbb{F}_{2^{36}}.$$

Group inversion on K is

$$\bar{a} = a^{2^{18}},$$

with fixed field $L = \mathbb{F}_q$, $q = 2^{18}$. On the nontrivial factor, (2.2) reduces to

$$D^2 + D = I. \tag{5.1}$$

For a primitive $\omega \in \mathbb{F}_4 \subseteq L$, one of the two shifts

$$E = D + \omega^2 I$$

is Hermitian and idempotent. Hence E is the orthogonal projection onto a nondegenerate subspace of the unitary space K^9.

Recall

$$|U(n,q)| = q^{n(n-1)/2} \prod_{i=1}^{n} (q^i - (-1)^i). \tag{5.2}$$

The finite unitary order formula, Witt geometry, and the associated transitivity and isotropic-subspace counts used below are standard; see, for example, Taylor's treatment of finite classical groups [10].

Theorem 5.1 (Characteristic-two census)**.** *For a fixed parity quotient and a fixed* $6/3$ *orientation, the nontrivial characteristic-two relaxation consists of exactly*

$$N_4 = \frac{|U(9,q)|}{|U(4,q)|\,|U(5,q)|}, \qquad q = 2^{18}, \tag{5.3}$$

rank-four Hermitian projections. This 217*-digit integer satisfies*

$$2^{719} < N_4 < 2^{720}. \tag{5.4}$$

The opposite orientation has rank five and the same count. Up to fiber permutation and complement, every integral lift in the complete quotient census injects into a union of at most two such sets; consequently

$$2^{720} < 2N_4 < 2^{721}. \tag{5.5}$$

Proof. Hermitian idempotents are in bijection with nondegenerate subspaces via their images. The unitary group is transitive on nondegenerate r-spaces, with stabilizer $U(r,q) \times U(9-r,q)$, giving (5.3). The trace law fixes $r = 4$ in one orientation and $r = 5$ in the other. Fixing both the trivial CRT factor and the nontrivial field factor uniquely fixes each binary word of length 37, so every integral lift injects into this relaxation. The parity collapse in Theorem 3.2 gives the factor two. □

The same calculation gives the characteristic-polynomial identity

$$\det(\lambda I + D(x)) = \lambda^9 + C(x)(\lambda^8 + \lambda^4 + 1) + \lambda^5 + \lambda \pmod 2, \tag{5.6}$$

where C is the oriented quadratic-class indicator. Equivalently, write

$$\det(\lambda I + D) = \sum_{i=0}^{9} e_i(D)\lambda^{9-i}, \qquad e_0(D) = 1,$$

so $e_i(D) \in \mathbb{F}_2[C_{37}]$ is the ith elementary symmetric coefficient, and let $\delta = x^0$ be the identity element of this group algebra. Then

$$e_1 = e_5 = e_9 = C, \qquad e_4 = e_8 = \delta, \qquad e_2 = e_3 = e_6 = e_7 = 0. \tag{5.7}$$

5.2 Characteristic three

Here

$$\mathbb{F}_3[C_{37}] \cong \mathbb{F}_3 \times \mathbb{F}_{3^{18}} \times \mathbb{F}_{3^{18}},$$

and inversion on either nontrivial factor is the unitary involution over $\mathbb{F}_q$, $q = 3^9$. If $N = D - I$, the nontrivial equation becomes

$$N^2 = 0, \qquad N^* = N. \tag{5.8}$$

Its image is totally isotropic, so $\operatorname{rank} N \leq 4$.

Let $I_{n,r}(q)$ denote the number of totally isotropic r-spaces in a nondegenerate n-dimensional Hermitian space:

$$I_{n,r}(q) = \prod_{j=0}^{r-1} \frac{(q^{n-2j} - (-1)^{n-2j})(q^{n-2j-1} - (-1)^{n-2j-1})}{q^{2(j+1)} - 1}. \tag{5.9}$$

Let

$$H_r(q) = \frac{|GL(r, q^2)|}{|U(r, q)|} \tag{5.10}$$

be the number of nonsingular Hermitian forms on a fixed r-space.

Theorem 5.2 (Characteristic-three census)**.** *For one degree-*18 *factor, the exact number of Hermitian square-zero matrices is*

$$M_3 = \sum_{r=0}^{4} I_{9,r}(3^9) H_r(3^9). \tag{5.11}$$

For a fixed trivial quotient, the two nontrivial factors are independent, so the relaxation contains M_3^2 points, with

$$2^{1141} < M_3^2 < 2^{1142}. \tag{5.12}$$

Proof. A Hermitian square-zero map is determined by its totally isotropic image U and the nonsingular Hermitian form induced on U by the map from the quotient dual to U. This gives the factor $I_{9,r}(q)H_r(q)$. The CRT split gives the square. □

Remark 5.3. Neither (5.4) nor (5.12) is an exact count of binary or integral conference graphs. These are separate upper bounds from different modular relaxations and must not be multiplied as if they were independent filters.

6 Exact characteristic-two physical supports

The unitary projection census does not itself impose the prescribed integer Hamming weights of all 45 cyclic blocks. Nevertheless both complement classes have an explicit exact-margin realization.

Theorem 6.1 (Two exact modular supports)**.** *For one integral quotient representing each of the two parity types up to complement, there exists a* 9×9 *matrix of binary cyclic words satisfying:*

(a) the prescribed exact integer margin of every block;

(b) star symmetry and zero diagonal identity coefficients;

(c) the $6/3$ *law* (4.1)*;*

(d) every one of the 2997 *ordered group-ring coefficients of*

$$D(x)^2 + D(x) = I_9 + U(x)J_9 \quad \text{over } \mathbb{F}_2. \tag{6.1}$$

Equivalently, the two expanded 333×333 *supports are loopless, have degree* 166*, and satisfy all* 110,889 *scalar equations.*

The text-witness SHA-256 values are

$$\begin{array}{l|l} \text{type 1} & \texttt{3f77fc3d39fc6b8dfd33efbba846e61ca835581d15f9ae4c12d8f57a3249e697} \\ \text{type 2} & \texttt{2e5087308dceb18398daaba4b1ca5868d755cecb41ae7892f988ae8083c03c22}. \end{array} \tag{6.2}$$

6.1 Construction outline

The CRT is particularly explicit. If $d \in K$ is represented by a polynomial of degree below 36 and $q_0 \in \mathbb{F}_2$ is the augmentation, the unique binary cyclic word is

$$f = d + c\Phi_{37}, \qquad c = q_0 + d(1). \tag{6.3}$$

A max-flow first prescribes a 9×18 incidence table for diagonal inverse pairs, simultaneously enforcing all nine margins and column sums six on residues and three on nonresidues.

Starting from a dense central projection, exact two-coordinate unitary rotations prescribe eight diagonal entries; the ninth is forced by trace. The off-diagonal margins are then imposed by diagonal unitary conjugation. The relevant unit circle has order

$$2^{18} + 1 = 262{,}145.$$

For each edge, all phase differences are enumerated, and the remaining nine-vertex difference constraint is solved exactly.

The following identity is the compatibility gate for off-diagonal norms.

Proposition 6.2 (Norm-support trace identity)**.** *Let* f *be a binary word of length* 37 *and weight* k*, evaluated at a nontrivial* 37*th root* ζ*. Then*

$$\operatorname{Tr}_{\mathbb{F}_{2^{18}}/\mathbb{F}_2}\big(f(\zeta)f(\zeta^{-1})\big) = \binom{k}{2} \pmod 2. \tag{6.4}$$

Proof. Expand the norm into parity autocorrelations. The 18 inverse-pair terms have absolute trace one because 2 is primitive modulo 37. Their sum counts each unordered pair of support positions once. $\square$

Remark 6.3. Theorem 6.1 concerns one integral representative of each parity type. It does not assert exact-margin realizability for all 625 integral quotients.

7 The characteristic-37 moment filtration

Work in

$$R = \mathbb{F}_{37}[y]/(y^{37}), \qquad x = 1 + y.$$

Since $U(x) = y^{36}$ and $18^2 + 18 = 9 \bmod 37$, set

$$N(y) = D(1 + y) - 18I_9.$$

Equation (2.2) becomes

$$N(y)^2 = 9y^{36}J_9. \tag{7.1}$$

Its constant term is

$$N_0 = B - 18I_9 = -\frac{T}{2} \pmod{37}, \tag{7.2}$$

a symmetric rank-four square-zero matrix annihilating $\mathbf{1}$.

7.1 The first moment

Star symmetry makes the first coefficient N_1 skew-symmetric. The coefficient of y in (7.1) is

$$N_0N_1 + N_1N_0 = 0. \tag{7.3}$$

Theorem 7.1 (Rank-16 first-moment law)**.** *For every quotient in Theorem 3.1, the linear system* (7.3) *on the* 36 *skew coordinates has rank* 16*, leaving a* 20*-dimensional kernel.*

For every fixed nontrivial block size, translation acts transitively on the possible first moments of a subset of $\mathbb{F}_{37}$*. Hence the* 16 *equations divide every fixed-margin ambient count by exactly*

$$37^{16}. \tag{7.4}$$

The 625*-class trace-law union therefore has logarithm*

$$1307.10873431446430 - 16\log_2(37) = 1223.75748046440094. \tag{7.5}$$

The first three nonconstant binomial moments can be realized simultaneously with the quotient margins, star symmetry, and the trace law for the certified quotient (3.1). The corresponding physical support satisfies (7.1) through degree three. Its first failure is degree four, with 79 of the 81 matrix entries nonzero. Thus several low layers are feasible but are not evidence of convergence to a graph.

7.2 A full formal completion

The correct coordinate for the involution is

$$z = \log(1 + y),$$

because cyclic reversal sends z to $-z$. In the truncated ring,

$$z^{36} = y^{36} \quad \text{and} \quad z^{18} = 6\sum_{t\neq 0}\chi(t)x^t, \tag{7.6}$$

where χ is the quadratic character of $\mathbb{F}_{37}$.

Theorem 7.2 (Formal integrability for the certified quotient)**.** *Let N_0 be* (7.2) *for the certified quotient* (3.1)*. Every admissible first moment*

$$X^T = -X, \qquad N_0 X + X N_0 = 0$$

has the form

$$X = [N_0, A]$$

for some symmetric $A \in M_9(\mathbb{F}_{37})$. For either $\eta \in \{1, -1\}$,

$$N_A(y) = \mathrm{e}^{-zA} \left(N_0 + \eta z^{18} J_9 + 19 y^{36} J_9 \right) \mathrm{e}^{zA} \tag{7.7}$$

has star symmetry and satisfies the full equation

$$N_A(y)^2 = 9 y^{36} J_9.$$

Proof. Exact elimination gives rank 20 for the map

$$\mathrm{Sym}_9(\mathbb{F}_{37}) \longrightarrow \mathrm{Skew}_9(\mathbb{F}_{37}), \qquad A \longmapsto [N_0, A],$$

equal to the dimension of the admissible first-moment kernel. Furthermore,

$$N_0^2 = 0, \qquad N_0 J = J N_0 = 0, \qquad J^2 = 9J.$$

The term $19y^{36}J$ lies in the terminal socle: it changes the trace but neither the square nor lower coefficients. Since $(\eta z^{18} J)^2 = z^{36} J^2 = 9 y^{36} J$, the middle matrix in (7.7) has the required square. Conjugation preserves the square and star symmetry. $\square$

Remark 7.3. Theorem 7.2 is not a classification of all formal solutions. More importantly, the coefficients of (7.7) need not be realizable simultaneously by binary cyclic blocks with the required margins. The theorem identifies physical support realizability, rather than formal matrix algebra, as the remaining obstruction in this chart.

8 Constant-generator obstructions through rank three

For each diagonal block, let $r_i(t)$ denote the coefficient after removing the common terminal $19y^{36}J$ contribution from (7.7). Since $y^{36} = U = \sum_t x^t$ and $D = N + 18I$, restoring the physical adjacency coefficient gives

$$D_{ii}(0) = r_i(0), \qquad D_{ii}(t) = r_i(t) + 19 \pmod{37} \quad (t \neq 0).$$

Thus the reduced target $\{18, 19\}$ below is equivalent to the restored adjacency target $\{0, 1\}$. A generic reduced cyclic word r must satisfy

$$r(0) = 0, \qquad r(t) \in \{18, 19\} \quad (t \neq 0). \tag{8.1}$$

Its number of nonzero-lag 19's is the diagonal block degree.

Theorem 8.1 (Constant low-rank obstruction)**.** *Within the family* (7.7)*:*

(a) no diagonal constant generator A is physical;

(b) no nonzero symmetric rank-one constant generator is physical;

(c) no symmetric rank-two constant generator is physical for any of the 625 *quotient classes;*

(d) no symmetric rank-three constant generator is physical for any of the 625 *quotient classes.*

Thus a viable single constant generator in this family must have rank at least four.

8.1 Ranks zero, one, and two

For diagonal A, every restored nonzero-lag coefficient of D is 13 or 25, rather than 0 or 1; these are not the reduced $\{18, 19\}$ values in (8.1).

Rank one separates into nonisotropic and isotropic cases. In the nonisotropic case, six explicitly occurring quadratic-character patterns force the normalized local parameters to satisfy $b_i = 0$ and $a_i \in \{3, -3\}$, while the rank-one parametrization gives $b_i = 0 \Rightarrow a_i = 1$. This contradiction is independent of the quotient.

In the isotropic case write $A = \lambda uu^T$ with $u^T u = 0$, so $A^2 = 0$. After the terminal $19y^{36}J$ term has been removed, every zero-lag diagonal correction is a multiple of z^k with $1 \leq k \leq 35$. Exact conversion from the z-basis to the cyclic x-basis gives

$$[x^0]z^k = 0 \qquad (1 \leq k \leq 35).$$

Consequently the reduced zero-lag coefficient is fixed, for every quotient and every coordinate, at

$$r_i(0) = N_{0,ii} = B_{ii} - 18 = -\frac{T_{ii}}{2} \pmod{37}.$$

Physicality would require $r_i(0) = 0$ for all i, hence $T_{ii} = 0$ for all i. The 625-class canonical census contains no quotient with all-zero diagonal, so no isotropic rank-one generator is physical for any quotient.

For rank two, split semisimple types are eliminated by an exhaustive local character test over all 35 projective eigenvalue ratios and all 37^2 projector-coordinate triples. The remaining rational types are:

(i) an irreducible quadratic block,
(ii) a nonzero J_2 block,
(iii) a nonzero eigenline plus $J_2(0)$,
(iv) $J_3(0)$,
(v) $J_2(0) \oplus J_2(0)$.

A similarity-invariant temporal overcode gives only the two Paley weight-18 words in cases (i), (iv), and (v). Cases (ii) and (iii) initially allow weights $16, \ldots, 20$, leaving one quotient sign-class with diagonal Seidel profile

$$(-4, -4, 0, 0, 0, 0, 0, 4, 4).$$

Restoring the fixed coefficient of $z^{18}J$ reduces the final diagonal condition to two affine cosets of a seven-dimensional cyclic code. The two genuine orientations $\eta = \pm 1$ have empty binary intersection, closing the exception.

8.2 Rank three

The rank-three audit uses a safe over-list of all projective rational similarity types, including types not realizable by symmetric matrices:

primary shape	projective types
invertible dimension three, cyclic	1371
invertible dimension three, noncyclic	37
$J_2(0)$ plus invertible dimension two	39
zero-primary rank two plus a nonzero line	2
pure nilpotent rank three	3
total	1452.

(8.2)

For a minimal polynomial of degree $m \le 4$, write

$$\mathrm{e}^{\pm zA} = \sum_{r=0}^{m-1} f_r^{\pm}(z) A^r.$$

Symmetry gives

$$\operatorname{diag}(A^r M A^s) = \operatorname{diag}(A^s M A^r)$$

for symmetric A, M. This yields a reduced temporal code

$$W_A = F_A + z^{18} F_A$$

spanned by the symmetrized products of the $f_r^{\pm}$. Its intersection with (8.1) contains at most 256 words for any type. Quotient diagonal profiles remove 492 types. For each of the remaining 960, the audit restores the fixed J coefficient and exhausts both trace orientations and at most $37^3 = 50{,}653$ relaxed local power tuples. No binary word remains. Every relaxation enlarges the candidate set, so emptiness proves the stated obstruction.

Remark 8.2. Theorem 8.1 does not exclude rank at least four, a genuinely nonconstant conjugator, a different formal parametrization, or the unrestricted 45-block lift.

9 The first nonconstant conjugator layer

For a matrix polynomial M, put

$$M^{\dagger} = M(-z)^T.$$

A near-identity conjugator G preserves star symmetry precisely when $G^{\dagger} = G^{-1}$. Since logarithm and exponential are inverse on the nilpotent ideal, write $G = \mathrm{e}^K$ with $K^{\dagger} = -K$. Thus

$$K = zA_0 + z^2 A_1 + z^3 A_2 + \cdots, \qquad A_r^T = (-1)^r A_r. \tag{9.1}$$

The first extension beyond a constant generator is

$$K = zA_0 + z^2 A_1, \qquad A_0^T = A_0, \quad A_1^T = -A_1. \tag{9.2}$$

Proposition 9.1 ($20 + 20$ normal form)**.** *Through degree two, gauge reduction of* (9.2) *leaves* 20 *essential parameters in* A_0 *and* 20 *in* A_1.

Proof. Expanding

$$\mathrm{e}^{-K} N_0 \mathrm{e}^{K} = N_0 + zX + z^2 Y + O(z^3)$$

gives

$$X = [N_0, A_0], \qquad Y = [N_0, A_1] + \frac{1}{2}[[N_0, A_0], A_0].$$

The exact commutator ranks are

$$\begin{aligned} \mathrm{Sym}_9 &\xrightarrow{[N_0,-]} \mathrm{Skew}_9 : \quad \operatorname{rank} = 20, \quad \dim\ker = 25, \\ \mathrm{Skew}_9 &\xrightarrow{[N_0,-]} \mathrm{Sym}_9 : \quad \operatorname{rank} = 20, \quad \dim\ker = 16. \end{aligned}$$

The homogeneous second tangent has dimension 21; its trace-zero subspace is exactly the second commutator image. Centralizer-valued left multiplication removes the two kernels, leaving the claimed slice. $\square$

Theorem 9.2 (Rank-two closures at the first nonconstant layer)**.** *Across all* 625 *quotient classes, neither of the following families can produce binary diagonal blocks:*

(a) the pure first-higher rank-two family $K = z^2 B$*, with* $B^T = -B$ *and* $\operatorname{rank} B = 2$*;*

(b) the complete family in which A_0 *and* A_1 *in* (9.2) *are supported on one common nondegenerate two-plane and* $A_1 \neq 0$*.*

9.1 Pure first-higher rank two

Write $B = uv^T - vu^T$. Then

$$B^3 = -\Delta B, \qquad \Delta = (u^T u)(v^T v) - (u^T v)^2,$$

so there are four projective rational types: split semisimple, irreducible trace-zero semisimple, $J_3(0)$, and $J_2(0) \oplus J_2(0)$. In every type, the exact diagonal temporal overcode meets the binary target only in the two Paley words, both of weight 18. Theorem 3.1 excludes an all-degree-18 quotient.

9.2 The common nondegenerate two-plane pencil

On the active plane, write

$$A_0 = tI + S, \qquad A_1 = B,$$

where $S^T = S$, $\operatorname{Tr} S = 0$, and $B^T = -B \neq 0$. Then

$$S^2 = \alpha I, \qquad B^2 = \beta I, \qquad SB + BS = 0, \quad \beta \neq 0. \tag{9.3}$$

The traceless exponent closes because

$$(zS + z^2 B)^2 = (\alpha z^2 + \beta z^4) I. \tag{9.4}$$

After decimation there are 76 orbits with $t = 0$ and 1332 normalized parameter types with $t \neq 0$. A symmetry-restored temporal overcode leaves only two exceptional nonzero-t types:

$$(\alpha, \beta) = (5, 32), \quad (\alpha, \beta) = (19, 20),$$

with allowed weight sets

$$\{14, 18, 18, 22\}, \qquad \{12, 18, 18, 24\},$$

respectively. Only the second meets any quotient profile, leaving four quotient classes and 40 trace-compatible overcode assignments.

For $(t, \alpha, \beta) = (1, 19, 20)$, the ordinary function space F and $z^{18}F$ are disjoint seven-dimensional spaces. Restoring the fixed $J = \mathbf{1}\mathbf{1}^T$ coefficient forces, at each coordinate, a ten-component vector determined by three scalars $(p, s, b) \in \mathbb{F}_{37}^3$. Exhausting all 37^3 relaxed triples for both $\eta = \pm 1$ gives zero survivors for every exceptional word. This closes the entire named pencil.

Remark 9.3. Degenerate common support with nonzero A_0, support that changes with degree, and nonconstant coefficients of rank at least three remain open.

10 The next 2-adic digit and small-switch gates

Throughout this section D denotes the expanded 333×333 binary adjacency matrix represented earlier by $D(x)$. Put

$$J = J_{333}, \qquad I = I_{333}, \qquad S = J - I - 2D.$$

Define the expanded carry

$$R(D) = \frac{D^2 + D - 83(I + J)}{2} \pmod 2. \tag{10.1}$$

Vanishing of $R(D)$ is exactly the adjacency equation modulo four. Since

$$S^2 - (333I - J) = 4\big(D^2 + D - 83(I + J)\big),$$

this is equivalently the conference-core equation modulo 16.

The two initial supports from Theorem 6.1 have, among the 1503 independent star-symmetric orbit coefficients of this expanded carry,

$$\begin{array}{c|cc} & \text{nonzero carries} & \text{zero carries} \\ \hline \text{type 1} & 722 & 781 \\ \text{type 2} & 764 & 739. \end{array} \tag{10.2}$$

A deterministic 200,000-move walk within one exact-margin type-1 phase fiber improves 722 to 672, consisting of 72 diagonal and 600 off-diagonal defects. This is an attained upper bound, not a minimum.

Theorem 10.1 (Exact small-switch gates)**.** *For each of the two original frozen supports:*

(a) no ordinary four-cycle toggle preserves the complete characteristic-two equation;

(b) among the 49,284 tested smallest semiregular Hermitian transvections, none preserves all exact block margins.

Proof. For a four-cycle toggle,

$$\Delta = uv^T + vu^T$$

with disjoint weight-two vectors u, v, and $\Delta^2 = 0$. The toggled matrix preserves $D^2 + D = I + J$ if and only if $\langle u, v\rangle$ is D-invariant. Exhausting all $\binom{333}{2} = 55{,}278$ pair-vectors gives minimum $\mathrm{wt}(Du) = 136$ and 138, respectively, and no invariant plane.

For the second family, work in $K = \mathbb{F}_2[x]/(\Phi_{37}(x))$ with $\bar{x} = x^{-1}$, and take

$$u = e_a + x^s e_b, \qquad a < b, \quad s \in \mathbb{F}_{37},$$

so $u^*u = 0$. The audited scalar family is explicitly

$$\mathcal{C}_{\text{small}} = \{1\} \cup \{x^t + x^{-t} : 1 \le t \le 18\} \cup \{1 + x^t + x^{-t} : 1 \le t \le 18\}.$$

Every $c \in \mathcal{C}_{\text{small}}$ satisfies $c = \bar{c}$, and

$$U_c = I_9 + c\,uu^*$$

is unitary. The audit exhausts the $36 \cdot 37 = 1332$ binomial isotropic directions and all 37 displayed scalars, hence 49,284 exact unitary conjugations per support. All remain loopless, and 1492 or 1604 retain the trace law, but none retains every block margin. □

These are exclusions of two finite named families, not of all support switches. A complete fixed-quotient CP-SAT model for the modulo-four problem has 1494 physical bits, 418,293 reused Boolean products, and 1503 independent cyclic congruences. Bounded runs returned `UNKNOWN`, including at the lower layer where a witness is known. Those solver outcomes are diagnostics of the formulation, not mathematical evidence.

11 Reproducibility

The mathematical source artifacts referenced below are contained in the repository directory

`hadamard_668_search/conference_334_z37_lift/`.

This paper package also includes the canonical census dump, its digest sidecar, and a dependency-free structural checker. The calculations use exact integer or finite-field arithmetic. No floating-point comparison, stochastic nonexistence inference, or timeout is used in a theorem.

Claim	Primary replay or certificate
Quotient census and 625-class dump	`QUOTIENT_CENSUS.md` and `census_z37_quotients.cpp`; separate parity audit `audit_z37_quotient_parity.cpp`; all-quotient ambient replay `audit_z37_all_quotient_ambient.py`; packaged `z37_quotient_census_canonical_625.txt`, `z37_quotient_census_canonical_625.sha256`, and `verify_package.py`.
Trace law, finite-field counts, first moment, and formal completion	`README.md`, `verify_z37_lift_frontier.py`, and `Z37_LIFT_FRONTIER_CERTIFICATE.json`.
Constant rank two	`RANK_TWO_CONJUGATION_OBSTRUCTION.md`, `RANK_TWO_JORDAN_OBSTRUCTION.md`, and the corresponding two verification scripts.
Constant rank three	`rank_three_constant/README.md`, `rank_three_constant/audit_rank_three_constant.py`, and `rank_three_constant/RANK_THREE_CONSTANT_CERTIFICATE.json`.
Two physical characteristic-two supports	`char2_support_realization/README.md`, `char2_support_realization/verify_all_char2_support.py`, `char2_support_realization/TYPE1_SUPPORT_WITNESS.json`, and `char2_support_realization/TYPE2_SUPPORT_WITNESS.json`.
First nonconstant normal form and closures	`first_nonconstant_conjugator/FIRST_NONCONSTANT_CONJUGATOR.md`, `first_nonconstant_conjugator/verify_first_nonconstant_gauge.py`, `first_nonconstant_conjugator/verify_exceptional_plane_fixed_j.py`, and `first_nonconstant_conjugator/FIRST_NONCONSTANT_CERTIFICATE.json`.
Four-cycle and transvection gates	`char2_carry_switches/README.md`, `char2_carry_switches/audit_four_cycle_and_carry_invariants.py`, and `char2_carry_switches/audit_sparse_unitary_transvections.py`.
Next 2-adic digit	`mod4_support_lift/README.md`, `mod4_support_lift/audit_char2_witness_mod4.py`, and `mod4_support_lift/audit_mod4_carry_chart.cpp`.

11.1 Minimal replay

The package-level certificate check uses only the Python standard library and is independent of the caller's working directory:

```
python3 hadamard_668_search/papers/semiregular_c37_conference/verify_package.py
```

From the repository root:

```
python3 hadamard_668_search/conference_334_z37_lift/verify_z37_lift_frontier.py
python3 hadamard_668_search/conference_334_z37_lift/
    verify_rank_two_conjugation_obstruction.py
```

```
python3 hadamard_668_search/conference_334_z37_lift/verify_rank_two_jordan_obstruction.py
python3 hadamard_668_search/conference_334_z37_lift/char2_support_realization/
    verify_all_char2_support.py
python3 hadamard_668_search/conference_334_z37_lift/first_nonconstant_conjugator/
    verify_first_nonconstant_gauge.py
python3 hadamard_668_search/conference_334_z37_lift/first_nonconstant_conjugator/
    verify_exceptional_plane_fixed_j.py
```

The exhaustive quotient replay is:

```
clang++ -O3 -std=c++20 \
  hadamard_668_search/conference_334_z37_lift/census_z37_quotients.cpp \
  -o /tmp/census_z37_quotients

/tmp/census_z37_quotients --dump-canonical \
  > /tmp/z37_625_canonical_final.txt
```

Its output must match the packaged dump with SHA-256

`c5d8765da49deb39c2ff3407b9d0f265e3ca56c1015d5b0075355c53ca60fb5b`.

The enumerator takes roughly 45 seconds and 60 MB on an Apple M1 Pro. The constant-rank-three fixed-J replay is the longest promoted single audit, taking about 5.5 minutes and 61 MB on the same machine.

The immutable archival location for this package is

`https://github.com/AlecKriebel/Math/releases/tag/h668-research-checkpoint-v1.0.0`.

12 Conclusion and open boundary

The semiregular C_{37} lane survives all quotient-level necessary conditions and survives characteristic two even after exact margins and the full 6/3 diagonal law are imposed. This is simultaneously positive and cautionary. It produces a sharply defined modular frontier, but it also shows that characteristic two is not an obstruction.

The stated bounded quotient system has 625 permutation classes, two parity types up to complement, and no zero-diagonal class. The finite-field relaxations and the characteristic-37 formal completion explain large parts of the algebra. Constant generators through rank three and two natural first-nonconstant rank-two families cannot realize the binary diagonal alphabet. Small local switching also fails on the two frozen modular supports.

What remains is not a flat search. The best retained support still has 672/1503 nonzero next-digit carries, and the normalized characteristic-two relaxation alone is larger than 2^{720}. Any successful continuation needs a global mechanism that couples physical margins with higher 2-adic carries, or a new obstruction that acts on the two unitary parity types.

We emphasize the exact endpoint:

> No conference graph or $H(668)$ is constructed. Neither the semiregular C_{37} lane nor unrestricted existence is excluded.

Computational and AI assistance

The derivations, programs, certificates, literature audit, and manuscript were developed with heavy assistance from ChatGPT 5.6 Sol under Alec Kriebel's direction. Alec Kriebel is the named human

author and describes himself as a non-specialist who cannot independently certify every derivation or implementation. All promoted claims are tied to explicit local artifacts and exact replay programs listed in Section 11. Those artifacts strengthen auditability; they do not replace expert review, establish priority, or constitute a proof-assistant formalization. No external communication was made in preparing this draft.

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