

The quantum chromatic number of the G_{19} join family

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Abstract

In Section 4.2 of *Quantum colorings of spheres*, Lalonde proposed the family of finite graphs $G_n = G_{19} \vee K_{n-3}$ as candidate witnesses for his complex-sphere obstruction. To avoid overloading the fixed base graph G_{19} , write J_n for his G_n . Lalonde proved $\chi_q^{(1)}(J_n) = n + 1$ for rank-one color projectors [2, Theorem 1.5] and conjectured the same value without a rank restriction. We prove that conjecture in the finite-dimensional projector model:

$$\chi_q(J_n) = n + 1 \quad (n \geq 3).$$

Consequently $\xi(J_n) = n < \chi_q(J_n) = n + 1$, confirming the candidate finite witnesses and giving an alternative finitary proof of Lalonde's complex-sphere lower bound. In fact, Lalonde's fixed-dimension and fixed-rank parameters also satisfy $\chi_q^{[d]}(J_n) = \chi_q^{(s)}(J_n) = n + 1$ for every $d, s \geq 1$. In particular, adjoining one apex to G_{19} gives a 20-vertex graph H with $\chi_q(H) = 5$.

The proof first symmetrizes an arbitrary coloring, so that every projector has a common rank r . For a fixed color, a rational sum-of-squares identity rigidifies vertices $1, \dots, 13$ into a rank- r sign-vector configuration. The remaining six subspaces are classified, including all non-transverse branches, by an r -plane invariant under a canonical complex structure. Orthogonality between two colors then preserves the two complex-structure sectors while reversing membership in the invariant planes. The resulting sector subspaces are pairwise orthogonal across colors, and their dimensions give the impossible inequality $3nr \leq 2nr$. The core sum-of-squares and all formal block identities have an exact, standard-library verifier.

1 Graph, model, and result

The vertex set of G_{19} is $\{1, \dots, 19\}$, and its edge set is

$$\begin{aligned} E_{19} = \{ & 12, 13, 18, 19, 1\,15, 23, 26, 27, 2\,14, 34, 35, 3\,16, 45, 4\,12, 4\,13, \\ & 5\,10, 5\,11, 5\,17, 67, 6\,11, 6\,13, 7\,10, 7\,12, 7\,19, 89, 8\,11, 8\,12, 8\,18, \\ & 9\,10, 9\,13, 14\,17, 14\,18, 15\,17, 15\,19, 16\,18, 16\,19 \}. \end{aligned}$$

Here vw denotes the unordered edge (v, w) . Its graph6 representation is

$$\text{RxLAKA@AgYAWDGO?O?@??A?W@@OC@_}.$$

The only triangles are

$$T_0 = 123, \quad T_1 = 189, \quad T_2 = 345, \quad T_3 = 267.$$

Here $\vee$ denotes graph join and K_0 is the empty graph. For $n \geq 3$, define

$$J_n = G_{19} \vee K_{n-3}.$$

Thus J_n is the graph denoted G_n in [2, Section 4.2]; the letter J merely avoids overloading G_{19} when $n = 19$. This is the family proposed there as a finite witness for the complex-sphere obstruction of [2, Theorem 1.2]. Here ξ denotes complex orthogonal rank. The first joined case is the 20-vertex one-apex graph

$$H = J_4 = G_{19} \vee K_1.$$

An m -coloring in the finite-dimensional projector model is a family of orthogonal projections $P_{v,c} \in M_d(\mathbb{C})$, with zero projections allowed, such that

$$\sum_{c=1}^m P_{v,c} = I_d, \quad P_{v,c} P_{w,c} = 0 \quad (vw \in E).$$

The row orthogonality $P_{v,c} P_{v,c'} = 0$ for $c \neq c'$ follows from, and may equivalently be included with, the assertion that the row is a projective partition of the identity. By the structure theorem recorded in [2, Theorem 2.19], following Cameron–Montanaro–Newman–Severini–Winter [1], this projector formulation is equivalent to the standard finite-dimensional definition of χ_q by perfect entangled strategies for the graph-coloring game. Following [2, Section 2.4.2], write $\chi_q^{[d]}$ for the chromatic number restricted to local dimension d , and $\chi_q^{(s)}$ for the rank- s quantum chromatic number.

Theorem 1.1 (Main theorem). *For every $n \geq 3$ and every $d, s \geq 1$,*

$$\xi(J_n) = n, \quad \chi_q(J_n) = \chi_q^{[d]}(J_n) = \chi_q^{(s)}(J_n) = \chi(J_n) = n + 1.$$

In particular, $\chi_q(H) = 5$.

For $n = 3$, the equality $\chi_q(J_3) = 4$ was already known: $J_3 = G_{19}$ contains the Mančinska–Roberson graph G_{13} and is classically four-colorable [2, Section 4.2]. Thus the new unrestricted cases begin with the one-apex graph J_4 ; the theorem is stated for all $n \geq 3$ because its proof is uniform.

The orthogonal-rank equality follows from $\xi(G_{19}) = 3$ [2, Theorem 1.6] and additivity under graph join [2, Theorem 2.17(5)]. For the elementary chromatic upper bound, Lalonde gives the four-coloring [2, Section 4.2]

$$(\kappa(1), \dots, \kappa(19)) = (1, 2, 3, 2, 1, 3, 1, 3, 2, 3, 2, 4, 1, 1, 2, 1, 3, 2, 3).$$

Giving the $n - 3$ joined vertices fresh colors proves $\chi(J_n) \leq n + 1$. The work is to exclude a quantum n -coloring. This also excludes every coloring with fewer than n colors, since unused colors may be padded by zero projections.

2 Clique saturation and uniformization

Lemma 2.1 (Clique columns). *Suppose an m -coloring contains an m -clique C . Then, for every color c ,*

$$\sum_{v \in C} P_{v,c} = I_d.$$

Thus the $m \times m$ array $(P_{v,c})_{v \in C, c \in [m]}$ is a magic unitary.

Proof. For fixed c , the summands are pairwise orthogonal, so their sum is a projection of trace at most d . Summing this trace bound over the m columns gives at most md . The row identities show that the total trace is exactly md , so equality holds in every column. A projection of trace d in $M_d(\mathbb{C})$ is the identity. $\square$

Lemma 2.2 (Color uniformization). *If J_n has a finite-dimensional quantum n -coloring, then it has one in a dimension $D = nr$ for which every projector has the same positive rank r .*

Proof. Given a coloring in dimension d , take its direct sum over the $n!$ color permutations:

$$\tilde{P}_{v,c} = \bigoplus_{\pi \in S_n} P_{v,\pi(c)}.$$

This is again a coloring. Its dimension is $D = n!d$, while

$$\text{rank } \tilde{P}_{v,c} = (n-1)! \sum_{a=1}^n \text{rank } P_{v,a} = (n-1)!d =: r.$$

Hence $D = nr$ and $r > 0$. □

Assume henceforth that such a uniform coloring exists. Fix a color c . Let R_c be the sum of the color- c projections on the joined clique K_{n-3} and let

$$Q_c = I_D - R_c.$$

The joined vertices are pairwise adjacent, so $\text{rank } R_c = (n-3)r$ and $\dim Q_c = 3r$. Every old-vertex projection $P_{v,c}$ is supported on Q_c . Applying Lemma 2.1 to $T_j \vee K_{n-3}$ gives

$$\sum_{v \in T_j} P_{v,c} = Q_c \quad (j = 0, 1, 2, 3). \quad (1)$$

Thus, on Q_c , the fixed-color projections form a uniform rank- r projective orthogonal representation of G_{19} in dimension $3r$.

3 Exact fixed-color rigidity

In this section fix the color and write $e_v = P_{v,c}|_{Q_c}$ and $I = I_{Q_c}$. All traces are the ordinary matrix trace on Q_c .

3.1 A rational core sum of squares

Put

$$X = e_{10} + e_{11} + e_{12} + e_{13}$$

and define the three Walsh forms

$$H_1 = e_{10} + e_{11} - e_{12} - e_{13},$$

$$H_2 = e_{10} - e_{11} + e_{12} - e_{13},$$

$$H_3 = e_{10} - e_{11} - e_{12} + e_{13}.$$

Also put $d_1 = e_5 - e_4$, $d_2 = e_7 - e_6$, and $d_3 = e_9 - e_8$.

Lemma 3.1 (Core SOS). *The defining projection, edge, and triangle relations imply the exact identity*

$$\frac{4}{3}I - X = \sum_{j=0}^3 f_j^* f_j, \quad (2)$$

where all f_j are self-adjoint and

$$f_0 = \frac{1}{2} \left(X - \frac{4}{3}I \right), \quad f_i = \frac{2}{3}d_i + \frac{1}{2}H_i \quad (1 \leq i \leq 3).$$

Proof. The triangle decompositions and edge zeros give

$$\sum_{i=1}^3 d_i^2 = (e_4 + e_5) + (e_6 + e_7) + (e_8 + e_9) = 2I. \quad (3)$$

Hadamard cancellation, together with $e_j^2 = e_j$, gives

$$X^2 + H_1^2 + H_2^2 + H_3^2 = 4X. \quad (4)$$

Finally,

$$\sum_{i=1}^3 \{d_i, H_i\} = -4X. \quad (5)$$

To see the last identity, collect the coefficient of each e_j for $j = 10, 11, 12, 13$. The corresponding signed sum of d_1, d_2, d_3 is $N_j - (2I - N_j) = 2N_j - 2I$, where N_j is the sum of the three projections among $e_4, \dots, e_9$ adjacent to j . Since $N_j e_j = e_j N_j = 0$, its anticommutator with e_j is $-4e_j$. Summing over j proves (5).

Expanding the right side of (2) and inserting (3)–(5) leaves exactly $\frac{4}{3}I - X$. $\square$

The two sides of (2) have the same trace, namely zero, because $\dim Q_c = 3r$ and each e_j has rank r . Faithfulness of the matrix trace therefore forces every f_j to vanish. Define

$$A = e_4 - e_5, \quad B = e_6 - e_7, \quad C = e_8 - e_9.$$

Walsh inversion now gives

$$\begin{aligned} e_{10} &= \frac{1}{3}(I + A + B + C), & e_{11} &= \frac{1}{3}(I + A - B - C), \\ e_{12} &= \frac{1}{3}(I - A + B - C), & e_{13} &= \frac{1}{3}(I - A - B + C). \end{aligned} \quad (6)$$

Moreover,

$$A^2 = I - e_3, \quad B^2 = I - e_2, \quad C^2 = I - e_1, \quad A^2 + B^2 + C^2 = 2I.$$

For each

$$(a, b, c) \in \{(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)\},$$

the idempotency of $\frac{1}{3}(I + aA + bB + cC)$ gives

$$ab\{A, B\} + ac\{A, C\} + bc\{B, C\} = aA + bB + cC.$$

Adding the $(1, 1, 1)$ equation to each of the other three in turn yields

$$\{B, C\} = A, \quad \{A, C\} = B, \quad \{A, B\} = C. \quad (7)$$

Lemma 3.2 (Core normal form). *There is a Hilbert space K_c of dimension r and a unitary $W_c : K_c^3 \rightarrow Q_c$ under which the ranges of $e_1, \dots, e_{13}$ are*

$$\begin{aligned} 1 &= (I, 0, 0), & 2 &= (0, I, 0), & 3 &= (0, 0, I), \\ 4 &= (I, I, 0), & 5 &= (I, -I, 0), \\ 6 &= (I, 0, I), & 7 &= (I, 0, -I), \\ 8 &= (0, I, I), & 9 &= (0, I, -I), \\ 10 &= (I, I, I), & 11 &= (I, I, -I), \\ 12 &= (I, -I, I), & 13 &= (I, -I, -I). \end{aligned}$$

Here a block row denotes the range of the associated block-column map from K_c ; normalization of the frame is immaterial to its range.

Proof. Use $e_1 + e_2 + e_3 = I$ to write $Q_c = K_1 \oplus K_2 \oplus K_3$, with $K_i = \text{ran } e_i$ and $\dim K_i = r$. The operator A is supported on $K_1 \oplus K_2$, B on $K_1 \oplus K_3$, and C on $K_2 \oplus K_3$. In the first relation of (7), the (1, 1) and (2, 2) blocks of $\{B, C\}$ vanish, so the two diagonal blocks of A vanish. Cyclically, all diagonal blocks vanish. Since $A^2 = I - e_3$, its (1, 2) block is unitary, and similarly for B, C . Thus

$$A = \begin{pmatrix} 0 & U & 0 \\ U^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & V \\ 0 & 0 & 0 \\ V^* & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & W \\ 0 & W^* & 0 \end{pmatrix}.$$

Equation (7) gives $U = VW^*$, $V = UW$, and $W = U^*V$. Set $K_c = K_1$ and define the unitary

$$G : K_c^3 \longrightarrow K_1 \oplus K_2 \oplus K_3, \quad G(x, y, z) = (x, U^*y, V^*z).$$

Pulling operators back by $M \mapsto G^*MG$ makes all three upper off-diagonal blocks the identity: they become $UU^* = I$, $VV^* = I$, and $UWV^* = I$. The binary decompositions and (6) then give the displayed ranges. $\square$

3.2 All tail branches

The tail is not always the graph of an operator; retaining degenerate planes is essential.

Lemma 3.3 (Tail-plane classification). *After the normalization in Lemma 3.2, the ranges at vertices 14, ..., 19 are determined by an r -plane $M_c \subset K_c \oplus K_c$ satisfying*

$$J_c M_c = M_c, \quad J_c = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \quad (8)$$

Explicitly,

$$\begin{aligned} 14 &= \{(a, 0, b) : (a, b) \in M_c\}, & 15 &= \{(0, a, b) : (a, b) \in M_c\}, \\ 16 &= \{(a, -b, 0) : (a, b) \in M_c\}, & & \\ 17 &= \{(x, x, z) : (x, z) \in M_c^\perp\}, & 18 &= \{(x, -z, z) : (x, z) \in M_c^\perp\}, \\ 19 &= \{(x, -z, x) : (x, z) \in M_c^\perp\}. \end{aligned} \quad (9)$$

Conversely, every J_c -invariant r -plane gives rank- r subspaces satisfying all fixed-color edge relations.

Proof. The six core–tail edges 2 14, 1 15, 3 16, 5 17, 8 18, and 7 19 place the tail ranges in the images of the following injective two-column frames:

v	$F_v(a, b)$	v	$F_v(a, b)$
14	$(a, 0, b)$	17	(a, a, b)
15	$(0, a, b)$	18	$(a, -b, b)$
16	$(a, -b, 0)$	19	$(a, -b, a)$.

Because each frame is injective, there is a unique r -plane $N_v \subset K_c^2$ with $\text{ran } e_v = F_v(N_v)$. The frames need not be isometries. For a tail edge vw ,

$$F_v(N_v) \perp F_w(N_w) \iff F_v^* F_w(N_w) \subseteq N_v^\perp,$$

so the cross-Gram pullbacks below require no normalization or transversality assumption. For the six tail–tail edges, direct pullback gives

vw	14 17	15 17	14 18	16 18	16 19	15 19
$F_v^* F_w$	I	I	I	I	I	$-J_c$.

Put $M_c = N_{14}$. The first five entries successively give $N_{17} = M_c^\perp$, $N_{15} = M_c$, $N_{18} = M_c^\perp$, $N_{16} = M_c$, and $N_{19} = M_c^\perp$. The final entry says $J_c M_c^\perp = M_c^\perp$. Because J_c is unitary and $J_c^* = -J_c$, this is equivalent to $J_c M_c = M_c$. The displayed frames are exactly (9). To make the complement equalities explicit, every edge first gives an orthogonality inclusion between two parameter planes. Each plane has dimension r in the same $2r$ -dimensional space, as does the orthogonal complement on the right, so every such inclusion is an equality. Reversing these equalities verifies the converse. $\square$

Remark 3.4. *If the first-coordinate projection of M_c is invertible, then $M_c = \text{ran}(I, S)$ and (8) is $S^2 = -I$. This transverse branch recovers the familiar operator-valued normal form. It is not exhaustive. For instance, $M = \text{span}\{(u, 0), (0, u)\} \subset \mathbb{C}^2 \oplus \mathbb{C}^2$, where $u = (1, 0)$, is J -invariant but is not a graph. Lemma 3.3 includes this branch. It is also not equivalent, under a unitary preserving the displayed labeled core, to an orthogonal direct sum of rank-one fixed-color representations. Indeed, under the sector parametrization*

$$M \cap \ker(J - \sigma i I) = \{(x, \sigma i x) : x \in E_\sigma\},$$

this example has $E_+ = E_- = \text{span}\{u\}$ and hence $\dim(E_+ \cap E_-) = 1$. Vertices 1, 2, 3 force every core-preserving unitary to be block diagonal, while vertices 4 and 6 force its three blocks to coincide; it is therefore $\text{diag}(U, U, U)$ and preserves this intersection dimension. By contrast, in a direct sum of rank-one representations, E_+ and E_- are orthogonal multiplicity summands. Thus fixed-color higher-rank representations are not all scalar direct sums, and an unrestricted proof must control these branches.

At $r = 1$, for $\sigma \in \{+1, -1\}$ every invariant line and its complement have the form

$$M_c = \text{span}\{(1, \sigma i)\}, \quad M_c^\perp = \text{span}\{(1, -\sigma i)\}.$$

Substitution into (9) gives

$$\begin{aligned} 14 &= (1, 0, \sigma i), & 15 &= (0, 1, \sigma i), & 16 &= (1, -\sigma i, 0), \\ 17 &= (1, 1, -\sigma i), & 18 &= (1, \sigma i, -\sigma i), & 19 &= (1, \sigma i, 1). \end{aligned}$$

Together with the core, this is exactly Lalonde's scalar normal form [2, Theorem 4.5] after the phase gauge $\alpha = \beta = 1$, with $b = \sigma$. Thus Lemmas 3.2 and 3.3 are a higher-rank extension of that normal form; their derivation does not invoke the computer-assisted nonidentification input [2, Lemma 4.4].

4 Cross-color sector obstruction

Take distinct colors c, d and use the core isometries from Lemma 3.2. Put

$$T_{dc} = W_d^* W_c : K_c^3 \longrightarrow K_d^3.$$

Lemma 4.1 (Cross-color block equations). *There are maps $X, Y, Z : K_c \rightarrow K_d$ such that*

$$T_{dc} = \begin{pmatrix} 0 & X & Y \\ -X & 0 & Z \\ -Y & -Z & 0 \end{pmatrix}. \quad (10)$$

For $L : K_c \rightarrow K_d$, define

$$\Omega_L = \begin{pmatrix} 0 & L \\ -L & 0 \end{pmatrix}.$$

Then, for each $L \in \{X, Y, Z\}$,

$$\Omega_L M_c \subseteq M_d^\perp, \quad \Omega_L M_c^\perp \subseteq M_d. \quad (11)$$

Proof. Same-vertex orthogonality at vertices 1, 2, 3 kills the diagonal blocks of T_{dc} . Vertices 4, 6, 8 make the corresponding lower block the negative of the upper block, giving (10). No adjoints occur here: T_{dc} maps the coordinate space for c to the one for d .

Compressing (10) by the two-column frames in (9), vertices 14, 15, 16 give, respectively,

$$\Omega_Y M_c \subseteq M_d^\perp, \quad \Omega_Z M_c \subseteq M_d^\perp, \quad -\Omega_X M_c \subseteq M_d^\perp.$$

Vertices 17, 18, 19 give

$$\Omega_{Y+Z} M_c^\perp \subseteq M_d, \quad \Omega_{Y-X} M_c^\perp \subseteq M_d, \quad \Omega_{Z-X} M_c^\perp \subseteq M_d.$$

The coefficient transform is invertible; for example

$$2X = (Y + Z) - (Y - X) - (Z - X),$$

with analogous formulas for Y, Z . This proves (11). $\square$

Let $\Gamma_c = 2P_{M_c} - I$ on K_c^2 . Equation (11) is equivalent to

$$\Gamma_d \Omega_L = -\Omega_L \Gamma_c \quad (L = X, Y, Z). \quad (12)$$

Because M_c reduces the unitary J_c , it splits across the $+i$ and $-i$ eigenspaces of J_c . Using the isometries up to a harmless factor

$$V_{c,\sigma}(x) = (x, \sigma i x), \quad \sigma \in \{+, -\},$$

write

$$M_c \cap \ker(J_c - \sigma i I) = V_{c,\sigma}(E_{c,\sigma}), \quad e_{c,\sigma} = \dim E_{c,\sigma}.$$

Then

$$e_{c,+} + e_{c,-} = r. \quad (13)$$

The operator Ω_L intertwines J_c with J_d and acts within a σ sector as the nonzero scalar σi times L . Combining this with (12) gives

$$L E_{c,\sigma} \subseteq E_{d,\sigma}^\perp \quad (c \neq d, L = X, Y, Z). \quad (14)$$

We can now finish the proof.

Proof of Theorem 1.1. Suppose that J_n had a quantum n -coloring. Uniformize it using Lemma 2.2, so the physical dimension is $D = nr$ and all ranks are $r > 0$. For each color c , apply Lemmas 3.2 and 3.3, and define the physical sector subspace

$$\mathcal{H}_{c,\sigma} = W_c(E_{c,\sigma} \oplus E_{c,\sigma} \oplus E_{c,\sigma}).$$

It has dimension $3e_{c,\sigma}$. Equations (10) and (14) show the pairwise orthogonality directly. For $c \neq d$, take $\xi = (\xi_1, \xi_2, \xi_3) \in E_{c,\sigma}^{\oplus 3}$ and $\eta = (\eta_1, \eta_2, \eta_3) \in E_{d,\sigma}^{\oplus 3}$. Then

$$\begin{aligned} \langle W_d \eta, W_c \xi \rangle &= \langle \eta, T_{dc} \xi \rangle \\ &= \langle \eta_1, X \xi_2 + Y \xi_3 \rangle + \langle \eta_2, -X \xi_1 + Z \xi_3 \rangle \\ &\quad + \langle \eta_3, -Y \xi_1 - Z \xi_2 \rangle = 0. \end{aligned}$$

Every term vanishes by (14): $\eta_j \in E_{d,\sigma}$, whereas $X\xi_k, Y\xi_k, Z\xi_k \in E_{d,\sigma}^\perp$. Thus, for fixed σ , the n subspaces $\mathcal{H}_{c,\sigma}$ are pairwise orthogonal in the common nr -dimensional physical space. Therefore

$$3 \sum_{c=1}^n e_{c,+} \leq nr, \quad 3 \sum_{c=1}^n e_{c,-} \leq nr.$$

Adding and using (13) gives

$$3nr = 3 \sum_{c=1}^n (e_{c,+} + e_{c,-}) \leq 2nr,$$

which is impossible because $n, r > 0$. Hence no quantum n -coloring exists. The classical $(n+1)$ -coloring in Section 1 gives $\chi_q(J_n) = \chi(J_n) = n+1$. Every fixed-dimension or rank- s coloring is finite-dimensional, giving the restricted lower bounds. Tensoring the classical solution with I_d gives the fixed-dimension upper bound; the direct sum of its $n+1$ cyclic relabelings, tensored with I_s , gives the fixed-rank upper bound. This explicit construction is needed because the rank- s chromatic numbers are not generally monotone in s , and adjoining an additional nonzero rank- s color is not automatic [2, Section 2.4.2]. Section 1's orthogonal-rank calculation completes the theorem. $\square$

Corollary 4.2 (Finitary complex-sphere obstruction). *For every $n \geq 3$, the complex unit sphere with orthogonality adjacency satisfies*

$$\chi_q(S_{\mathbb{C}}^{n-1}) \geq n+1.$$

In particular, Theorem 1.1 gives an alternative finitary proof of [2, Theorem 1.2].

Proof. The equality $\xi(J_n) = n$ gives a graph homomorphism $J_n \rightarrow S_{\mathbb{C}}^{n-1}$, and the quantum chromatic number is monotone under graph homomorphisms [2, Sections 1.1 and 2.3.3]. Theorem 1.1 therefore gives

$$n+1 = \chi_q(J_n) \leq \chi_q(S_{\mathbb{C}}^{n-1}).$$

This finite witness does not follow merely from the infinitary lower bound by a general de Bruijn–Erdős principle: χ_q does not have that property [2, Proposition 3.3]. $\square$

5 Relation to the rank-one theorem and exact verification

The three higher-rank obstacles. The closing discussion of [2, Section 4.2] identifies three barriers to the unrestricted problem: rank-one propagation no longer classifies Grassmannian-valued representations; no evident analogue of Lemma 4.6 supports the rank-one proof; and even a few values of n appear to require very large, n -specific SOS programs. Lemmas 3.1–3.3 address the first barrier by classifying every fixed-color higher-rank representation through one rational core SOS and a J -invariant tail plane. Lemma 4.1, together with the subsequent sector relations, replaces the role of Lemma 4.6 by the structure needed across colors. These statements are independent of n , so one exact rational identity and the final packing argument treat the entire family without an SDP.

Remark 3.4 shows that the higher-rank classification recovers Lalonde's scalar normal form [2, Theorem 4.5], but the proof does not assume that classification or equal original ranks. Symmetrization makes fixed-color ranks uniform, the SOS classifies the fusion-frame core, and Lemma 3.3 treats every tail, including non-graph and noncommuting branches. The obstruction therefore covers arbitrary finite dimension, zero projectors, and reducible or irreducible representations. On the

induced Mančinska–Roberson graph G_{13} , operator propagation proving $\chi_q(G_{13}) = 4$ [4] is an earlier higher-rank rigidity precedent; the present argument uses the extra six-vertex tail. For context, the computer-assisted method behind Lalonde’s Lemma 4.4 is developed in [3]; neither that lemma nor its algorithm is used in the present classification.

Two standard-library obstruction verifiers replay the SOS, signs, six tail compressions, J intertwining, final symbolic dimension arithmetic, and an independent trace-SOS formulation with clique and cross-color checks. A separate graph checker verifies graph6, no four-clique, Lalonde’s four-coloring, and exhaustive non-three-colorability. All arithmetic is exact; no numerical SDP enters. The programs do not formalize the unitary gauge step in Lemma 3.2, the projection/subspace semantics and complement equalities in Lemma 3.3, or the final orthogonal-subspace packing; those remain part of the human proof.

Artifact provenance. The certificate payloads and original algebraic audit were frozen in the initial, pre-presentation immutable project snapshot with full Git object name

`b1944a23707eb69d2f9f25eda0bb73c32cd5500a.`

Relative to that project root, the certificate payloads have SHA-256 digests

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certificate/obstruction_certificate.json:
fdf2aff0df2902b08ad0363e6448e299c7b04d1437b329634c9d7a719376c76c
certificate/lalonde_uniform_obstruction.json:
e5e0a6126c1adc4088e090ea58ca194f8c2f0ce243f9cd9c4c151329e97a71e1
```

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The current paper, technical summary, source, certificates, and verification instructions are available at <https://aleckriebe1.github.io/Math/papers/lalonde-quantum-coloring/>.

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