Technical summary: the G_{19} join family

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Result. Lalonde's Section 4.2 family is $G_n = G_{19} \vee K_{n-3}$; write it as J_n to avoid overloading the fixed 19-vertex base graph. The base graph has graph6 string `RxLAKA@AgYAWDGO?O?@??A?W@@OC@_`. For every $n \geq 3$ and $d, s \geq 1$, with ξ denoting complex orthogonal rank,

$$\boxed{\xi(J_n) = n < \chi_q(J_n) = n+1,} \qquad \chi_q^{[d]}(J_n) = \chi_q^{(s)}(J_n) = n+1.$$

This confirms Lalonde's candidate finite witnesses, removes the rank-one restriction, and gives an alternative finite-witness proof that $\chi_q(S_{\mathbb{C}}^{n-1}) \geq n+1$. The $n=3$ equality was already known; the new unrestricted cases begin with the one-apex graph $H = J_4$, for which $\chi_q(H) = 5$. Zero projectors, nonuniform original ranks, arbitrary finite dimension, and noncommuting projectors are all allowed.

Uniformization and clique saturation. Suppose J_n has a quantum n-coloring in dimension d. Direct-sum over all color permutations:

$$\widetilde{P}_{v,c} = \bigoplus_{\pi \in S_n} P_{v,\pi(c)}.$$

The new dimension is $D = n!d = nr$, and every projector has common positive rank $r = (n-1)!d$. Thus uniformity is a theorem-level reduction, not an ansatz. For an n-clique, each fixed-color column consists of pairwise orthogonal projections. Its trace is at most D; summing over all colors and using the row partitions makes all n inequalities equalities. Hence every clique column sums to I_D.

Fix a color c. Let R_c be the sum of its projections on the joined clique and $Q_c = I - R_c$. Each of 123, 189, 345, and 267, joined with K_{n-3}, is an n-clique. Therefore

$$\dim Q_c = 3r, \qquad \sum_{v \in T_j} P_{v,c} = Q_c.$$

On Q_c, write $e_v = P_{v,c}|_{Q_c}$. These are rank-r projections giving a projective orthogonal representation of G_{19}.

Exact core SOS. Set $X = e_{10} + e_{11} + e_{12} + e_{13}$,

$$\begin{aligned} H_1 &= e_{10} + e_{11} - e_{12} - e_{13}, \\ H_2 &= e_{10} - e_{11} + e_{12} - e_{13}, \qquad (d_1, d_2, d_3) = (e_5 - e_4, e_7 - e_6, e_9 - e_8). \\ H_3 &= e_{10} - e_{11} - e_{12} + e_{13}, \end{aligned}$$

Projection, edge, and triangle relations give

$$\sum_i d_i^2 = 2I, \qquad X^2 + \sum_i H_i^2 = 4X, \qquad \sum_i \{d_i, H_i\} = -4X.$$

The final identity is graph-specific: for each $j = 10, \ldots, 13$, the positive Walsh signs select exactly the three neighbors of j among $4, \ldots, 9$. Expanding the following factors now gives the exact rational identity

$$\frac{4}{3}I - X = \sum_{j=0}^{3} f_j^* f_j, \quad f_0 = \frac{1}{2}\left(X - \frac{4}{3}I\right), \quad f_i = \frac{2}{3}d_i + \frac{1}{2}H_i.$$

Both sides have trace zero, since $\dim Q_c = 3r$ and each e_j has rank r. Faithfulness of finite matrix trace forces every $f_j = 0$.

Put $A = e_4 - e_5$, $B = e_6 - e_7$, and $C = e_8 - e_9$. Walsh inversion and $e_j^2 = e_j$ yield

$$\{B, C\} = A, \qquad \{A, C\} = B, \qquad \{A, B\} = C.$$

Relative to $Q_c = \operatorname{ran} e_1 \oplus \operatorname{ran} e_2 \oplus \operatorname{ran} e_3$, support restrictions put A, B, C on coordinate pairs $12, 13, 23$. The anticommutator equations kill their diagonal blocks; the remaining blocks are unitaries with one cocycle relation. A block-diagonal gauge makes all three blocks I_r. Thus vertices $1, \ldots, 13$ are exactly the scalar sign-vector core tensored with a multiplicity space $K_c \cong \mathbb{C}^r$.

Exhaustive tail. The remaining ranges are encoded by an arbitrary r-plane $M_c \subset K_c \oplus K_c$ invariant under $J_c = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$:

$$\begin{array}{c|ccc} v & 14 & 15 & 16 \\ \hline \operatorname{ran} e_v & (a,0,b) & (0,a,b) & (a,-b,0) \end{array} \quad (a,b) \in M_c,$$

$$\begin{array}{c|ccc} v & 17 & 18 & 19 \\ \hline \operatorname{ran} e_v & (x,x,z) & (x,-z,z) & (x,-z,x) \end{array} \quad (x,z) \in M_c^{\perp}.$$

The six core–tail edges supply the displayed ambient frames. Pulling back the six tail–tail edges gives five identity maps and one $-J_c$, proving both exhaustiveness and $J_c M_c = M_c$. This formulation includes non-graph planes; $M_c = \operatorname{ran}(I, S_c)$ with $S_c^2 = -I$ is only a transverse chart. For $r = 2$, the plane $\operatorname{span}\{(u, 0), (0, u)\}$ has coincident sign-sector parameter spaces, an invariant which shows that it is not core-gauge-equivalent to a direct sum of scalar representations. At $r = 1$, $M_c = \operatorname{span}(1, \sigma i)$ and the six displayed frames recover Lalonde's Theorem 4.5 normal form with sign $b = \sigma$.

Cross-color sector packing. Choose a core unitary $W_c : K_c^3 \to Q_c$. For $c \neq d$, same-vertex orthogonality at vertices $1, \ldots, 9$ forces

$$W_d^* W_c = \begin{pmatrix} 0 & X & Y \\ -X & 0 & Z \\ -Y & -Z & 0 \end{pmatrix}.$$

The lower blocks are negatives, not adjoints, because the matrix maps the c coordinate space to the d coordinate space. With $\Omega_L = \left(\begin{smallmatrix} 0 & L \\ -L & 0 \end{smallmatrix}\right)$, the six tail compressions are

$$\Omega_Y,\ \Omega_Z,\ -\Omega_X,\ \Omega_{Y+Z},\ \Omega_{Y-X},\ \Omega_{Z-X}.$$

Their invertible coefficient transform implies, for $L = X, Y, Z$,

$$\Omega_L M_c \subseteq M_d^{\perp}, \qquad \Omega_L M_c^{\perp} \subseteq M_d.$$

Since M_c reduces J_c, write

$$M_c \cap \ker(J_c - \sigma i I) = \{(x, \sigma i x) : x \in E_{c,\sigma}\}, \quad e_{c,+} + e_{c,-} = r.$$

Each Ω_L intertwines J_c and J_d, so the plane flip says $L E_{c,\sigma} \subseteq E_{d,\sigma}^{\perp}$. For fixed σ, the n physical spaces

$$W_c(E_{c,\sigma} \oplus E_{c,\sigma} \oplus E_{c,\sigma})$$

are pairwise orthogonal in dimension nr. Hence

$$3 \sum_c e_{c,+} \leq nr, \qquad 3 \sum_c e_{c,-} \leq nr.$$

Adding gives $3nr \leq 2nr$, impossible for $r > 0$. Thus no quantum n-coloring exists. The explicit classical four-coloring of G_{19} and fresh colors on the joined clique give the matching $(n+1)$ upper bound.

Verification. Two standard-library exact obstruction verifiers and a separate graph checker accompany the proof. They decode the graph6 checksum, verify the published four-coloring and exhaustive non-three-colorability, replay independent rational noncommutative SOS expansions, check the Walsh and skew-block systems, multiply all six tail compressions, invert their coefficient transform, and check the symbolic all-n dimension gap. No floating-point computation enters the theorem. The gauge, subspace/complement semantics, and final packing remain human linear algebra rather than formalized steps.

Scope and relation to prior work. The known rank-one theorem motivates the sign frame, but it is not used as a reduction: the SOS proves the module-valued core directly, and the invariant plane analysis handles genuinely higher-rank tails. The contradiction excludes every finite-dimensional representation of the n-coloring relations, reducible or irreducible. No finite local dimension supports n colors. At $n + 1$ colors, the classical solution lifts to fixed dimension d by tensoring with I_d and to fixed rank s by taking the direct sum of its cyclic color relabelings and tensoring with I_s. No claim is made about infinite-dimensional or commuting-operator colorings. The proof answers the three obstacles identified at the end of Lalonde's Section 4.2: it classifies all fixed-color higher-rank branches, replaces the role of Lemma 4.6 by a plane-flip lemma, and works for every n with one rational SOS rather than large n-specific SDPs.