

Minimum Bell-Setting Complexity for Qubit POVM–PVM Separation

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Abstract

Vértesi and Bene exhibited a 3×2 Bell scenario in which qubit positive-operator-valued measurements (POVMs) outperform qubit projection-valued measurements (PVMs). We prove the matching setting lower bound. For arbitrary finite, input-dependent output alphabets, the shared-randomness convex hulls of bipartite qubit-POVM and qubit-PVM behaviors are equal when both parties have two inputs. Equivalently, no two-input linear Bell functional separates the two convexified sets. We also give a rational 3×2 functional, an exact qubit-POVM strategy, and a global analytic qubit-PVM upper bound with a strict exact gap. Hence 3×2 is the minimum input architecture, up to exchanging the parties. The equality proof combines a one-binary-party Lorentz-cone simulation, an extremal binary–ternary reduction, a physically exact Lorentz incidence manifold, finite-POVM duality, and a rank trichotomy based on the incidence second variation. The result concerns convexified behavior sets; it does not assert equality of the raw nonconvex strategy images or same-state simulation.

Main Theorem. *For every finite input-dependent output architecture $(A_0, A_1; B_0, B_1)$,*

$$\text{conv } Q_2^{\text{POVM}}(\mathbf{A}, \mathbf{B}) = \text{conv } Q_2^{\text{PVM}}(\mathbf{A}, \mathbf{B}).$$

For the rational 3×2 Bell functional in equation (1),

$$\beta_{\text{POVM}} \geq L_0 := 20\sqrt{2} + \frac{16}{25} > U := 20\sqrt{2} + \frac{3}{5} + \frac{4 + 3\sqrt{2}}{250} \geq \beta_{\text{PVM}}.$$

Consequently, up to exchanging the parties, three inputs for one party and two for the other are necessary and sufficient for a strict fixed-qubit POVM–PVM separation by a linear Bell functional.

Throughout, “ $m \times n$ ” counts Bell inputs (measurement settings), not measurement outcomes.

1 Introduction

A positive-operator-valued measurement (POVM) is more general than a projection-valued measurement (PVM) on a fixed Hilbert space. Naimark dilation represents a POVM projectively only after adding degrees of freedom, so it does not preserve a fixed local dimension. Bell experiments therefore offer an operational question: how many measurement inputs are needed before generalized measurements on local qubits generate correlations unavailable to projective qubit measurements?

This belongs to the broader study of Bell nonlocality and dimension-constrained quantum behavior [1, 2].

Gisin asked whether some Bell inequality and quantum state can require nonprojective measurements for optimal violation, noting that binary outcomes cannot furnish such an example [3]. Vértesi and Bene answered that existence question for two qubits using three inputs on Alice and two on Bob [4]; subsequent work refined that family [5], and related device-independent constructions certify nonprojective qubit measurements with larger input architectures [6]. These results do not settle the two-input behavior-set question.

Two recent results approach nearby boundaries from different directions. Cerf and Ollivier develop a second-order perturbative method for $(n, 2, d)$ Bell scenarios near deterministic correlations and propose, without resolving, a route to distinguish raw fixed-dimensional POVM and PVM correlation sets [7]. Their proved qubit result concerns local dichotomic geometry and neither establishes nor implies the global, convexified arbitrary-output equality proved here. Panahi *et al.*, using an intermediate reduction attributed to Masanes together with Schmidt compression, show that when one party has two binary settings, every bipartite convexly extremal quantum correlation admits a two-qubit realization even if the other party has arbitrary settings and outcomes [8]. That is a Hilbert-space dimension theorem, not a projectivization of the arbitrary-output measurements. The one-binary-party theorem below instead supplies a common PVM mixture of the complete fixed-qubit behavior.

To our knowledge, equality of the shared-randomness-convexified fixed-qubit POVM and PVM behavior sets for all bipartite two-input architectures with finite, input-dependent output alphabets, and the resulting 3×2 minimum-input classification, were not previously known. It also differs from operator-level projective simulability of individual POVMs [9, 10]: a Bell behavior may admit a projective realization using a different state even when the original local POVM is not itself a mixture of PVMs.

Strong reductions are known when every measurement is dichotomic [11, 12]; genuine ternary effects are precisely what those arguments do not address. They also make exhaustive PVM conventions essential. A published ternary-measurement claim was later corrected after a qubit PVM with a zero effect was found to attain the same value [13, 14]. We therefore retain zero projectors, deterministic inputs, every support pattern, and stochastic postprocessing throughout.

Contributions

We work throughout with local Hilbert-space dimension at most two and with shared-randomness convexification. Our results are:

1. An explicit rational 3×2 Bell functional has an exact algebraic qubit-POVM strategy above a global analytic upper bound for all qubit-PVM strategies (theorem 3.1).
2. If one party has binary outcomes on both of its two inputs, every qubit POVM behavior is a common convex mixture of complete qubit-PVM behaviors (theorem 4.2).
3. Any remaining strict two-input separator reduces to a pure entangled strategy with one binary PVM and one ternary rank-one POVM per party (theorem 4.5).
4. The residual architecture cannot strictly exceed the PVM support function (theorem 9.3). The proof treats every smooth and singular rank of an exact Lorentz incidence model.
5. Consequently 3×2 is the minimum input architecture, up to exchanging the parties (corollary 9.4).

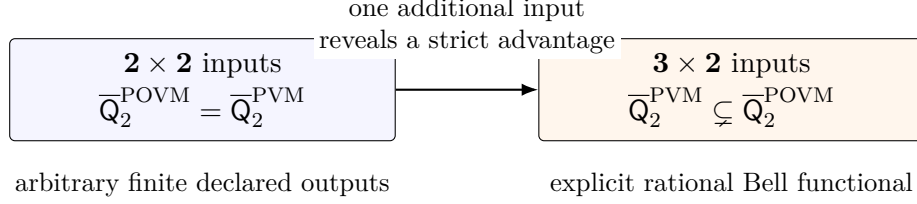


Figure 1: The minimum input threshold at fixed local dimension two. Equality means equality after shared-randomness convexification for each fixed finite declared-output architecture.

Table 1: The earlier 3×2 separator and the present exact classification. Both use the heterogeneous output architecture $(2, 2, 3)$ -by- $(2, 2)$.

Feature	Vértesi–Bene [4]	Present work
Role	Establishes fixed-qubit POVM advantage at 3×2	Proves arbitrary-output convexified equality at 2×2 , then combines it with an exact 3×2 witness
Bell coefficients	Algebraic I_{CH3} family	Rational functional
Unrestricted-state PVM comparison	Numerical optimization supported by an NPA upper bound for the displayed parameter choice	Analytic upper bound covering every two-qubit state and every PVM support
Global optimum claimed	No	No; L_0 is a lower bound and U an upper bound
Setting minimality	Not addressed	3×2 , up to party exchange

The equality is not an operator identity between POVMs and PVMs, nor does it assert simulation on the same state. It is an equality of convex behavior sets. The distinction is central.

Proof roadmap

The exact separation combines a CHSH locking term with a three-state discrimination task. For the equality theorem, convex separation first turns any counterexample into a global Bell maximizer. A Lorentz-cone circuit argument handles every case in which one party is binary. A common-span filtering perturbation and extremal-POVM dimension bound reduce all other counterexamples to one binary and one ternary measurement per party.

In the residual architecture, the five local effects form a unique binary-versus-ternary circuit. Lorentz coordinates describe the complete strategy by a 4×4 probability block P and a four-parameter metric g , subject to five quadratic null equations. Finite POVM duality makes every determinant multiplier strictly positive at a hypothetical strict separator. The constrained second variation then has ambient inertia $(4, 12)$. If the metric differential has rank at least two, dimension alone forces an uphill direction. Rank one contradicts positivity through an exact projective-fiber theorem, while rank zero has an explicit deterministic decomposition.

2 Model and convexified behavior sets

Let $X = Y = \{0, 1\}$ unless stated otherwise. Each input has a finite declared output alphabet A_x or B_y , which may depend on the input.

Definition 2.1 (Fixed-qubit behavior). A single fixed-qubit POVM strategy consists of Hilbert

spaces $\mathcal{H}_A, \mathcal{H}_B$ of dimensions at most two, a density operator ρ on $\mathcal{H}_A \otimes \mathcal{H}_B$, and effects

$$M_a^x \geq 0, \quad \sum_{a \in A_x} M_a^x = \mathbb{I}_A, \quad N_b^y \geq 0, \quad \sum_{b \in B_y} N_b^y = \mathbb{I}_B.$$

It produces

$$p(a, b | x, y) = \text{Tr}[\rho(M_a^x \otimes N_b^y)].$$

The resulting set is denoted $\mathbf{Q}_2^{\text{POVM}}(\mathbf{A}, \mathbf{B})$.

A PVM strategy requires all effects in each measurement to be mutually orthogonal projections. Zero projections are permitted. Therefore a qubit PVM with arbitrarily many declared labels has at most two nonzero effects. No ancillary system or Naimark dilation may increase a local dimension above two.

We denote the shared-randomness hulls by

$$\overline{\mathbf{Q}}_2^{\text{POVM}}(\mathbf{A}, \mathbf{B}) := \text{conv } \mathbf{Q}_2^{\text{POVM}}(\mathbf{A}, \mathbf{B}), \quad \overline{\mathbf{Q}}_2^{\text{PVM}}(\mathbf{A}, \mathbf{B}) := \text{conv } \mathbf{Q}_2^{\text{PVM}}(\mathbf{A}, \mathbf{B}).$$

Local stochastic postprocessing is included on the PVM side.

The raw sets $\mathbf{Q}_2^{\text{POVM}}$ and $\mathbf{Q}_2^{\text{PVM}}$ are strategy images and need not be convex: a single strategy has no external shared-randomness register. Their barred versions are convex hulls and do permit a common classical variable selecting the state and all local measurements simultaneously. For a Bell tensor c , define the support values

$$\beta_X(c) := \max_{p \in \overline{\mathbf{Q}}_2^X} \langle c, p \rangle, \quad X \in \{\text{POVM}, \text{PVM}\}.$$

Linearity gives the same maximum over the corresponding raw set, but equality of all support values determines only the convex hulls, not the raw sets.

Lemma 2.2 (Convexification and postprocessing). *For a linear Bell functional, shared randomness and stochastic postprocessing do not increase the optimum beyond one deterministic projective component.*

Proof. Linearity gives

$$\mathcal{F}\left(\sum_{\lambda} q_{\lambda} p_{\lambda}\right) = \sum_{\lambda} q_{\lambda} \mathcal{F}(p_{\lambda}) \leq \max_{\lambda} \mathcal{F}(p_{\lambda}).$$

A finite stochastic map is a convex combination of deterministic maps. Absorb all deterministic choices into the shared variable. A deterministic map merges mutually orthogonal projectors assigned to the same displayed label and inserts zero projectors for unused labels, hence remains a declared- output PVM. $\square$

For fixed finite alphabets, all strategy spaces above are compact. Indeed, density matrices and tuples of effects form closed bounded sets in finite-dimensional matrix spaces, and the Born map is continuous. The convex hull of a compact subset of a finite-dimensional affine space is compact: affine dependence reduces every convex combination to at most $m + 1$ points in affine dimension m .

Lemma 2.3 (Nearest-point separation). *Let $\mathcal{P}$ be a compact convex subset of a finite-dimensional Euclidean space. If $q \notin \mathcal{P}$ and p_0 minimizes $\|q - p\|$ over $p \in \mathcal{P}$, then $c = q - p_0$ satisfies*

$$c \cdot q > \sup_{p \in \mathcal{P}} c \cdot p.$$

Proof. For $p \in \mathcal{P}$, differentiate $\|q - p_0 - t(p - p_0)\|^2$ at $t = 0^+$. Minimality gives $(q - p_0) \cdot (p - p_0) \leq 0$, and strictness follows from $q \neq p_0$. $\square$

Proposition 2.4 (Convex hulls and Bell support functions). *For fixed finite alphabets,*

$$\overline{\mathbf{Q}}_2^{\text{POVM}} = \overline{\mathbf{Q}}_2^{\text{PVM}} \iff \beta_{\text{POVM}}(c) = \beta_{\text{PVM}}(c) \text{ for every real Bell tensor } c.$$

This equivalence makes no claim that $\mathbf{Q}_2^{\text{POVM}} = \mathbf{Q}_2^{\text{PVM}}$.

Proof. Equality of the convex sets implies equality of their maxima under every linear functional. Conversely, if the compact convex sets differed, a point in one but not the other would, by lemma 2.3, yield a linear functional with unequal maxima. Finally, a linear functional has the same maximum on a compact set and on its convex hull. $\square$

Remark 2.5 (Notation for arbitrary outputs). The shorthand $\overline{\mathbf{Q}}_2^{\text{POVM}}(2, 2) = \overline{\mathbf{Q}}_2^{\text{PVM}}(2, 2)$ means an equality schema: the equality holds separately for every choice of finite $(A_0, A_1; B_0, B_1)$. It is not an identification of behavior vectors living in different ambient spaces.

3 An exact rational 3×2 separation

For this section take Alice inputs $x = 0, 1, 2$, Bob inputs $y = 0, 1$, Alice labels $a = 0, 1, 2$, and Bob labels $b = 0, 1$. Define signs

$$\alpha_0 = \alpha_2 = 1, \quad \alpha_1 = -1, \quad \beta_0 = 1, \quad \beta_1 = -1$$

and correlators

$$E_{xy} = \sum_{a,b} \alpha_a \beta_b p(a, b|x, y).$$

For $x = 0, 1$, Alice's labels 0 and 2 have identical coefficients and may be deterministically coarse-grained. Thus the common three-label notation is equivalent to the heterogeneous architecture $(2, 2, 3)$ -by- $(2, 2)$. Set

$$\begin{aligned} \mathcal{B}(p) = & 10(E_{00} + E_{01} + E_{10} - E_{11}) \\ & + \frac{3}{5}p(0, 0|2, 0) + \frac{3}{5}p(1, 1|2, 0) + \frac{4}{5}p(2, 0|2, 1). \end{aligned} \tag{1}$$

For this declared-output architecture, write

$$\beta_{\text{POVM}}(\mathcal{B}) := \sup_{p \in \overline{\mathbf{Q}}_2^{\text{POVM}}} \mathcal{B}(p), \quad \beta_{\text{PVM}}(\mathcal{B}) := \sup_{p \in \overline{\mathbf{Q}}_2^{\text{PVM}}} \mathcal{B}(p).$$

Theorem 3.1 (Exact fixed-qubit separation). *For the rational functional (1),*

$$\beta_{\text{POVM}}(\mathcal{B}) \geq L_0 := 20\sqrt{2} + \frac{16}{25} > U := 20\sqrt{2} + \frac{3}{5} + \frac{4 + 3\sqrt{2}}{250} \geq \beta_{\text{PVM}}(\mathcal{B}).$$

The certified gap is

$$L_0 - U = \frac{3(2 - \sqrt{2})}{250} > 0.$$

These are an attained POVM lower bound and a global PVM upper bound; neither is asserted to be the corresponding exact optimum.

3.1 The simple exact strategy

Let X, Z be Pauli matrices and

$$|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}.$$

Bob uses $B_0 = Z$, $B_1 = X$. Alice uses

$$A_0 = (Z + X)/\sqrt{2}, \quad A_1 = (Z - X)/\sqrt{2}$$

for her first two binary measurements, with a zero third label. Her final measurement is

$$M_0 = (17\mathbb{I} - 8X + 15Z)/50,$$

$$M_1 = (17\mathbb{I} - 8X - 15Z)/50,$$

$$M_2 = (16\mathbb{I} + 16X)/50.$$

The rank-one factorizations

$$M_0 = \frac{1}{25} \begin{pmatrix} 4 \\ -1 \end{pmatrix} (4, -1), \quad M_1 = \frac{1}{25} \begin{pmatrix} 1 \\ -4 \end{pmatrix} (1, -4), \quad M_2 = \frac{8}{25} \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1, 1)$$

prove positivity. Direct addition of the displayed matrices gives

$$M_0 + M_1 + M_2 = \mathbb{I},$$

which proves normalization separately. Their nonzero eigenvalues are $17/25, 17/25, 16/25$, so the measurement is not a PVM.

The identity

$$\langle \Phi^+ | R \otimes S | \Phi^+ \rangle = \frac{1}{2} \text{Tr}(R^T S)$$

gives CHSH value $2\sqrt{2}$ and

$$p(0, 0|2, 0) = p(1, 1|2, 0) = p(2, 0|2, 1) = \frac{8}{25}.$$

Thus the simpler value

$$L_0 = 20\sqrt{2} + \frac{16}{25}$$

already exceeds U , since

$$L_0 - U = \frac{3(2 - \sqrt{2})}{250} > 0.$$

At the ideal point the auxiliary term is a state-discrimination problem with

$$W_0 = \frac{3}{20}(\mathbb{I} + Z), \quad W_1 = \frac{3}{20}(\mathbb{I} - Z), \quad W_2 = \frac{1}{5}(\mathbb{I} + X).$$

The dual operator

$$\Gamma = \frac{8}{25}\mathbb{I} + \frac{2}{25}X$$

satisfies $\Gamma \geq W_i$ and $(\Gamma - W_i)M_i = 0$, so the POVM value is $16/25$. Explicitly,

$$\Gamma - W_0 = \frac{1}{50} \begin{pmatrix} 1 \\ 4 \end{pmatrix} (1, 4), \quad \Gamma - W_1 = \frac{1}{50} \begin{pmatrix} 4 \\ 1 \end{pmatrix} (4, 1),$$

$$\Gamma - W_2 = \frac{3}{25} \begin{pmatrix} 1 \\ -1 \end{pmatrix} (1, -1),$$

so dual feasibility is positive-semidefinite by inspection. Every one- or two-label qubit PVM support scores at most $3/5$. The ideal discrimination advantage is therefore $1/25$.

3.2 Global projective bound

Write $\mathcal{B} = 10S + T$. For a PVM strategy, the displayed CHSH observables are Hermitian unitaries, including possible values $\pm \mathbb{I}$. If one is degenerate, the CHSH identity

$$C_{\text{CHSH}}^2 = 4\mathbb{I} - [A_0, A_1] \otimes [B_0, B_1]$$

gives $S \leq 2$, while $T \leq 7/5$, which is below U .

Otherwise optimize the state to a pure state and put it in Schmidt form

$$|\psi_\eta\rangle = \sqrt{\frac{1+\eta}{2}}|00\rangle + \sqrt{\frac{1-\eta}{2}}|11\rangle, \quad 0 \leq \eta \leq 1.$$

Its correlation matrix is

$$R = \text{diag}(\sqrt{1-\eta^2}, -\sqrt{1-\eta^2}, 1).$$

If $u = |\mathbf{b}_0 \cdot \mathbf{b}_1|$ and $\delta = 2\sqrt{2} - S$, optimizing Alice's directions first gives

$$S \leq \|R(\mathbf{b}_0 + \mathbf{b}_1)\| + \|R(\mathbf{b}_0 - \mathbf{b}_1)\|.$$

The normalized sum and difference directions are orthogonal. Since the two largest eigenvalues of $R^T R$ sum to $2 - \eta^2$, Cauchy-Schwarz gives

$$S \leq 2\sqrt{2 - \eta^2} \leq 2\sqrt{2} - \frac{\eta^2}{\sqrt{2}}.$$

The last comparison follows by squaring two nonnegative sides; their squared difference is $\eta^4/2$.

Independently, R is a contraction, so

$$\begin{aligned} S &\leq \sqrt{2+2u} + \sqrt{2-2u} \\ &= \sqrt{4+4\sqrt{1-u^2}} \leq \sqrt{8-2u^2} \leq 2\sqrt{2} - \frac{u^2}{2\sqrt{2}}. \end{aligned}$$

Here $\sqrt{1-u^2} \leq 1 - u^2/2$ and $\sqrt{4-u^2} \leq 2 - u^2/4$; the exact squared differences and their nonnegative right sides are recorded in appendix A. Hence

$$S \leq 2\sqrt{2} - \frac{\eta^2}{\sqrt{2}}, \quad S \leq 2\sqrt{2} - \frac{u^2}{2\sqrt{2}}, \quad (2)$$

hence

$$\eta \leq 2^{1/4}\sqrt{\delta}, \quad u \leq 2^{3/4}\sqrt{\delta}.$$

A ternary qubit PVM has one of the six labeled rank patterns obtained by permuting $(2, 0, 0)$ or $(1, 1, 0)$. For every singleton support and the pair $\{0, 1\}$, $T \leq 3/5 + 2(\eta + u)/5$. A pair involving label 2 reduces to binary discrimination between

$$\Pi(\mathbf{v}) := \frac{1}{2}(\mathbb{I} + \mathbf{v} \cdot \boldsymbol{\sigma}), \quad D = \text{diag}\left(\sqrt{\frac{1+\eta}{2}}, \sqrt{\frac{1-\eta}{2}}\right).$$

Writing $\tilde{\mathbf{b}}_y$ for the reflection of $\mathbf{b}_y$ induced by matrix transposition, the three score operators are

$$K_0 = \frac{3}{5}D\Pi(\tilde{\mathbf{b}}_0)D, \quad K_1 = \frac{3}{5}D\Pi(-\tilde{\mathbf{b}}_0)D, \quad K_2 = \frac{4}{5}D\Pi(\tilde{\mathbf{b}}_1)D.$$

216 The reflection preserves z -coordinates, dot products, and u . In a pair involving label 2, write the
 217 two selected score operators as

$$218 \quad G = \frac{3}{5}D\Pi(\mathbf{n})D, \quad H = \frac{4}{5}D\Pi(\mathbf{m})D.$$

219 With $x = n_z$, $y = m_z$, and $r = \mathbf{n} \cdot \mathbf{m}$,

$$220 \quad 20F_\eta = 7 + \eta(3x + 4y) + \sqrt{[-1 + \eta(3x - 4y)]^2 + 24(1 - \eta^2)(1 - r)}.$$

221 The exact comparison

$$222 \quad F_\eta \leq F_0(r) + \frac{2\eta}{5} \leq \frac{3}{5} + \frac{2}{5}(\eta + u)$$

223 is proved by the manifestly nonnegative polynomial certificate in appendix A. Consequently

	Nonzero labels	Rank pattern	Bound used for T
	$\{0\}$	$(2, 0, 0)$	$\text{Tr } K_0 \leq 3(1 + \eta)/10$
	$\{1\}$	$(0, 2, 0)$	$\text{Tr } K_1 \leq 3(1 + \eta)/10$
224	$\{2\}$	$(0, 0, 2)$	$\text{Tr } K_2 \leq 2(1 + \eta)/5$
	$\{0, 1\}$	$(1, 1, 0)$	$\text{Tr}(K_0 + K_1) \leq 3/5$
	$\{0, 2\}$	$(1, 0, 1)$	$F_\eta \leq 3/5 + 2(\eta + u)/5$
	$\{1, 2\}$	$(0, 1, 1)$	$F_\eta \leq 3/5 + 2(\eta + u)/5$

225 Each of the first four displayed estimates is no larger than the common bound $3/5 + 2(\eta + u)/5$.
 226 Therefore

$$227 \quad T \leq \frac{3}{5} + \frac{2}{5}(\eta + u) \leq \frac{3}{5} + \kappa\sqrt{\delta}, \quad \kappa = \frac{2}{5}(2^{1/4} + 2^{3/4}),$$

228 where $\kappa^2 = (16 + 12\sqrt{2})/25$. Completing the square,

$$229 \quad \mathcal{B} \leq 20\sqrt{2} + \frac{3}{5} + \kappa\sqrt{\delta} - 10\delta \leq 20\sqrt{2} + \frac{3}{5} + \frac{\kappa^2}{40} = U.$$

230 This covers every state, support pattern, direction, relabeling, and degeneracy. Lemma 2.2 covers
 231 shared randomness and postprocessing. A stronger algebraic strategy and its exact gap comparison
 232 are given separately in appendix B.

233 4 Two-input reductions

234 4.1 One binary party

235 **Lemma 4.1** (Circuits of a three-dimensional pointed cone). *Let K be a pointed convex cone of real*
 236 *dimension at most three. Consider a support-minimal equality between positive sums of nonzero*
 237 *extreme-ray vectors of K , after canceling any ray that occurs on both sides. Positive coefficients*
 238 *may be absorbed into the ray vectors. Then either the equality is a one-versus-one cancellation on*
 239 *the same ray, or K has dimension three and the relation has the form*

$$240 \quad r_0 + r_1 = s_0 + s_1$$

241 *with four distinct rays. In particular, every nontrivial circuit is two-versus-two.*

Proof. Move the two sums to opposite sides and attach the corresponding positive and negative coefficients. Support minimality makes this a minimally supported linear dependence. Indeed, if a proper subset admitted any nonzero dependence, add a sufficiently small signed multiple of it to the original coefficient vector and vary the multiple until one coefficient first vanishes; all original signs remain weakly unchanged, yielding a smaller positive relation after cancellation. A minimally dependent set in dimension d has at most $d + 1$ vectors, so at most four rays occur.

Pointedness excludes a dependence with all signs equal. If one side contained a single extreme ray and the other side contained at least two different rays, division by its positive coefficient would express that extreme ray as a nontrivial positive sum of other cone rays, a contradiction. Thus a nontrivial relation contains at least two rays on each side and at most four total, forcing $d = 3$ and the displayed two-versus-two form. A pointed cone of dimension one or two has at most two extreme rays, so only one-versus-one, same-ray cancellations remain there. A zero vector is not an extreme ray and is simply omitted. $\square$

Theorem 4.2 (One-binary-party simulation). *Suppose one party in a two-input bipartite qubit scenario has binary declared outputs on both inputs. The other party may have arbitrary finite outputs. Then every qubit-POVM behavior belongs to the convex hull of fixed-qubit PVM behaviors. The statement is symmetric under exchanging the parties.*

Proof. Decompose each binary qubit POVM into PVMs. If E has eigenvalues $0 \leq \mu \leq \lambda \leq 1$ and P is the spectral projector for λ , then

$$\{E, \mathbb{I} - E\} = (\lambda - \mu)\{P, \mathbb{I} - P\} + \mu\{\mathbb{I}, 0\} + (1 - \lambda)\{0, \mathbb{I}\}.$$

Taking the product of the decompositions for both inputs reduces, in each shared-random component, Bob to fixed binary PVMs with observables B_0, B_1 .

For Alice outcome $a|x$, let $\sigma_{a|x}$ be Bob's subnormalized steered state and define

$$\Phi_B(\sigma) = (\text{Tr } \sigma, \text{Tr}(\sigma B_0), \text{Tr}(\sigma B_1)) \in \mathbb{R}^3.$$

These data determine all joint probabilities:

$$p(a, b|x, y) = \frac{1}{2} \left[(\Phi_B \sigma_{a|x})_0 + (-1)^b (\Phi_B \sigma_{a|x})_{y+1} \right].$$

No-signaling gives equality of the sums of the two compressed ensembles.

If $\mathbb{I}, B_0, B_1$ are linearly independent, the image of the positive qubit cone is a three-dimensional Lorentz cone: the cone over the elliptical projection of the Bloch ball onto the span of the two Bloch axes. Parallel, antiparallel, or scalar observables give a two-dimensional simplicial cone or a local deterministic degeneration; these cases are included in appendix C.

Decompose every compressed positive vector into extreme rays, retaining its original output label. Repeatedly extract a support-minimal positive subrelation and subtract the largest multiple that leaves every residual coefficient nonnegative. At least one coefficient vanishes at each step, so this circuit elimination terminates. By lemma 4.1, every nontrivial circuit is

$$r_0 + r_1 = s_0 + s_1.$$

The original positive coefficients have been absorbed into the displayed ray vectors. One-versus-one, repeated-ray, zero-ray, and lower-dimensional cases produce only the degenerate components described below.

Every extreme compressed ray has a unique positive rank-one lift inside $\text{span}_{\mathbb{R}}\{\mathbb{I}, B_0, B_1\}$. Thus the circuit lifts to

$$R_0 + R_1 = S_0 + S_1 = \Omega$$

between rank-one positive operators. If Ω is invertible, the two rank-one decompositions become two orthonormal bases after applying $\Omega^{-1/2}$. Define Alice's two binary PVMs by

$$P_i^{(0)} = \left(\Omega^{-1/2} R_i \Omega^{-1/2}\right)^T, \quad P_i^{(1)} = \left(\Omega^{-1/2} S_i \Omega^{-1/2}\right)^T \quad (i = 0, 1).$$

Under the vectorization convention used here, the normalized canonical purification

$$|\Psi_{\Omega}\rangle = (\mathbb{I} \otimes \sqrt{\Omega / \text{Tr } \Omega})(|00\rangle + |11\rangle)$$

obeys

$$\text{Tr}_A \left[(P_i^{(0)} \otimes \mathbb{I}) |\Psi_{\Omega}\rangle \langle \Psi_{\Omega}| \right] = \frac{R_i}{\text{Tr } \Omega}, \quad \text{Tr}_A \left[(P_i^{(1)} \otimes \mathbb{I}) |\Psi_{\Omega}\rangle \langle \Psi_{\Omega}| \right] = \frac{S_i}{\text{Tr } \Omega}.$$

The transpose is required because $\text{Tr}_A[(P \otimes \mathbb{I})|\Omega_2\rangle\langle\Omega_2|] = P^T$ for $|\Omega_2\rangle = |00\rangle + |11\rangle$.

Rank-one Ω , zero terms, and repeated rays require separate bookkeeping. If $\Omega = |\omega\rangle\langle\omega|$ has rank one, every nonzero summand is collinear with Ω , so

$$R_i = \rho_i \Omega, \quad S_i = \sigma_i \Omega, \quad \rho_0 + \rho_1 = \sigma_0 + \sigma_1 = 1.$$

Put $|\hat{\omega}\rangle = |\omega\rangle / \sqrt{\text{Tr } \Omega}$ and use the product state $|0\rangle \otimes |\hat{\omega}\rangle$. For input 0, take the first projector onto $\sqrt{\rho_0}|0\rangle + \sqrt{\rho_1}|1\rangle$; for input 1, take the first projector onto $\sqrt{\sigma_0}|0\rangle + \sqrt{\sigma_1}|1\rangle$. The complementary projectors give the second outcomes. This realizes both arbitrary binary weight vectors. Zero terms receive zero projectors. Collinear terms are combined only within their retained output labels; if the same geometric ray occurs under different labels, classical splitting restores those labels branchwise. Thus no outcome probability is lost when repeated rays are removed.

Give this circuit strategy shared weight $\text{Tr } \Omega$. Circuit elimination preserves total trace, so the weights sum to one. The retained output labels and fixed Bob PVMs reconstruct every original compressed vector and hence every joint probability. The shared variable records the binary-POVM decomposition choices, the circuit component, and any repeated-label split. Conditional on that single variable it fixes a state, both Alice PVMs, and both Bob PVMs simultaneously. This is one common decomposition of the complete behavior, not a separate decomposition for each input. $\square$

4.2 Extreme residual architecture

Lemma 4.3 (Extreme points of a compact hull). *If K is compact in a finite-dimensional real vector space, every extreme point of $\text{conv } K$ belongs to K .*

Proof. Every point of $\text{conv } K$ is a finite convex combination of points of K . If an extreme point p had a representation containing two distinct positive-weight points, separating one term from the normalized sum of the others would express p as a nontrivial convex combination. Hence every positive-weight point in the representation equals p , and $p \in K$. $\square$

Lemma 4.4 (Common-span filtering). *Let Alice's two POVMs have a common Hermitian combination*

$$H = \sum_a s_{a|0} M_{a|0} = \sum_a s_{a|1} M_{a|1}.$$

For sufficiently small $\varepsilon > 0$, define

$$S_{\pm} = \mathbb{I} \pm \varepsilon H, \quad r_{a|x}^{\pm} = 1 \pm \varepsilon s_{a|x},$$

$$M_{a|x}^{\pm} = r_{a|x}^{\pm} S_{\pm}^{-1/2} M_{a|x} S_{\pm}^{-1/2}.$$

With

$$q_{\pm} = \text{Tr}[(S_{\pm} \otimes \mathbb{I})\rho], \quad \rho_{\pm} = \frac{(S_{\pm}^{1/2} \otimes \mathbb{I})\rho(S_{\pm}^{1/2} \otimes \mathbb{I})}{q_{\pm}},$$

the original complete behavior satisfies

$$p = \frac{q_+}{2} p_+ + \frac{q_-}{2} p_-.$$

Proof. The effects normalize because $\sum_a r_{a|x}^{\pm} M_{a|x} = S_{\pm}$, and $q_+ + q_- = 2$. Direct cyclicity of trace gives

$$q_{\pm} p_{\pm}(a, b|x, y) = r_{a|x}^{\pm} p(a, b|x, y).$$

Averaging the signs proves the identity for all inputs and outputs. $\square$

Theorem 4.5 (Residual-architecture reduction). *If a Bell functional with two inputs per party has a strict qubit-POVM advantage over $\overline{\mathcal{Q}}_2^{\text{PVM}}$, a global maximizing component may be chosen with:*

- (i) a pure full-Schmidt-rank two-qubit state;
- (ii) one nondegenerate binary PVM on each party;
- (iii) one extremal ternary rank-one POVM on each party;
- (iv) local real measurement spans intersecting only in $\mathbb{RI}$.

In particular, four-outcome extremal POVMs are unnecessary.

Proof. Choose an extreme point p of the exposed maximizing face. Because an extreme point of a face is extreme in the whole convex set, lemma 4.3 supplies one unrandomized raw strategy realizing p .

Pure state. Decompose its state into finitely many pure states while holding all measurements fixed. The resulting behaviors form a convex decomposition of p , so extremality forces every positive-weight behavior to equal p . A product component is local and cannot attain a strict value above the PVM support; choose a pure entangled component. Its two reduced states have full rank.

Extremal measurements. For a fixed finite outcome alphabet, the POVM set is compact and convex in a finite-dimensional affine space. An elementary induction on the dimension of its smallest containing face shows that it is the convex hull of its extreme points: if a point is not extreme, extend a nontrivial line segment through it to the relative boundary in both directions, then apply the induction to the two proper faces containing the endpoints. Affine dependence reduces the resulting representation to a finite one. Apply such a decomposition to one input at a time while fixing the state and all other measurements. Again every positive-weight behavior equals p , so an extremal component may be retained for every input. Temporarily delete zero effects; they contribute no probabilities and their declared output labels can be reinserted at the end. Full-rank reduced states make every remaining outcome marginal strictly positive. For one party set

$$V_x = \text{span}_{\mathbb{R}}\{M_{a|x}\} \subset \text{Herm}(2).$$

If $V_0 \cap V_1$ contained a nonscalar H , lemma 4.4 would split the behavior. Equality of both branch behaviors would imply, using strictly positive outcome marginals, $r_{a|x}^\pm = q_\pm$ for every nonzero outcome, making all coefficients equal and H scalar. Thus

$$V_0 \cap V_1 = \mathbb{R}\mathbb{I}.$$

Since $\dim_{\mathbb{R}} \text{Herm}(2) = 4$, one of V_0, V_1 has dimension at most two. Every two-dimensional real operator system containing $\mathbb{I}$ has the form $\text{span}_{\mathbb{R}}\{\mathbb{I}, H\}$ for a nonscalar Hermitian H . All of its elements commute. If $H = h_+P_+ + h_-P_-$ is the spectral decomposition, every effect $M_a = \alpha_a\mathbb{I} + \beta_aH$ equals

$$M_a = q(a|+)P_+ + q(a|-)P_-,$$

where $q(a|\pm) = \alpha_a + \beta_a h_\pm \geq 0$ and $\sum_a q(a|\pm) = 1$. Thus the POVM is a stochastic postprocessing of the binary spectral PVM $\{P_+, P_-\}$.

Deterministic postprocessing. A finite stochastic map is a convex combination of deterministic maps. It therefore decomposes p into behaviors obtained from deterministic relabelings of the spectral PVM. Extremality retains one component realizing p ; merging spectral projections and adding zero labels is again a PVM. Apply lemma 4.4 once more to that realization, so its two final spans still intersect only in $\mathbb{R}\mathbb{I}$. The PVM is nondegenerate: if it were deterministic, that party would have only one nontrivial input, and the behavior would be local.

The other span has dimension at most three. Its nonzero effects are linearly independent; otherwise lemma 4.4 with $H = 0$ splits the extreme behavior. One nonzero effect would be $\mathbb{I}$, a deterministic PVM. An extremal two-effect POVM is a PVM: diagonalizing one effect and perturbing either eigenvalue strictly between 0 and 1 would decompose it, so its eigenvalues are 0 or 1. A genuine nonprojective residual measurement therefore has at least three nonzero effects, while linear independence in a three-dimensional span gives at most three.

For an extremal qubit POVM, support-preserving perturbations give

$$\sum_a \text{rank}(M_a)^2 \leq 4.$$

The standard extremality criterion [15, 16] is reproved in appendix D for completeness. With three nonzero effects, any rank-two effect would make the rank-square sum at least $4 + 1 + 1 > 4$; hence all three effects have rank one. If both spans on one party were at most two, theorem 4.2 would make the full behavior PVM-simulable. Apply the argument to both parties, then reinsert any deleted zero-output labels. $\square$

5 Lorentz incidence model

Write every Hermitian 2×2 matrix as

$$X = x_0\mathbb{I} + x_1\sigma_x + x_2\sigma_y + x_3\sigma_z$$

and set $J = \text{diag}(1, -1, -1, -1)$. Then

$$\det X = x^T J x, \quad \text{Tr}(XY) = 2x^T y,$$

where the second pairing is the ordinary Euclidean coordinate pairing. Nonzero positive rank-one effects are precisely future-null vectors: $x_0 > 0$, $x^T J x = 0$.

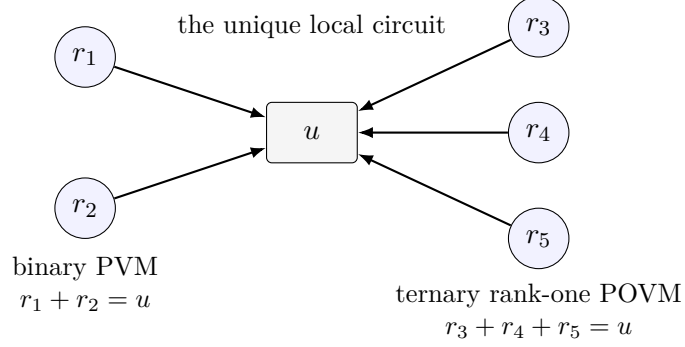


Figure 2: The local coefficient circuit after choosing four effects as an operator basis. The physical effects are future-null rays in a Lorentz metric.

In the residual architecture label one party's two binary effects by E_1, E_2 and ternary effects by E_3, E_4, E_5 . Set

$$r_1 = e_1, \quad r_2 = e_2, \quad r_3 = e_3, \quad r_4 = e_4, \quad r_5 = (1, 1, -1, -1)^T, \quad u = (1, 1, 0, 0)^T.$$

Their unique coefficient circuit is

$$r_1 + r_2 = r_3 + r_4 + r_5 = u.$$

The first four physical effects form a real basis of $\text{Herm}(2)$. Indeed, E_1, E_2 span the binary operator system V_0 , while the linearly independent ternary effects span V_1 . Since $V_0 \cap V_1 = \mathbb{R}\mathbb{I}$, the quotient $V_1/\mathbb{R}\mathbb{I}$ has dimension two. The images of any two ternary effects are independent there: a dependence modulo $\mathbb{I} = E_3 + E_4 + E_5$ would give a linear dependence among all three ternary effects. After relabeling those two effects as E_3, E_4 , the four operators E_1, E_2, E_3, E_4 are a basis.

Let E_A contain their Lorentz vectors as columns and define $g = E_A^T J E_A$. Nullness and normalization force

$$g = \begin{pmatrix} 0 & \frac{1}{2} & a & b \\ \frac{1}{2} & 0 & c & d \\ a & c & 0 & e \\ b & d & e & 0 \end{pmatrix}, \quad e = a + b + c + d - \frac{1}{2}. \quad (3)$$

On the genuine residual stratum all distinct-ray products are positive; in particular $a, b, c, d, e > 0$ and

$$a + b, c + d, a + c, b + d < \frac{1}{2}.$$

Definition 5.1 (Strict residual domain). A metric g of the form (3) is *strict residual* when it has signature $(1, 3)$, $u^T g u = 1$, and the five g -null coefficient rays are distinct and lie in one chosen future cone. Equivalently for this normal form, all products between distinct rays are positive, which includes

$$a, b, c, d, e > 0, \quad a + b, c + d, a + c, b + d < \frac{1}{2}.$$

The genuine residual domain additionally requires P invertible, $u^T P u = 1$, all five incidences below, strictly positive marginals $u^T P r_j$, and a full-rank reconstructed reduced state. These conditions exclude zero effects, repeated rays, product states, and loss of time orientation; all are open conditions around a genuine residual strategy.

Lemma 5.2 (Lorentz representation of a residual strategy). *Choose the first four effects on both parties as bases, with coordinate frames $\mathbf{E}_A, \mathbf{E}_B$, and let P be their 4×4 joint-probability block. For a normalized full-Schmidt-rank pure state*

$$|\psi_\Xi\rangle = \sum_{m,n=0}^1 \Xi_{mn} |mn\rangle, \quad \text{Tr}(\Xi^\dagger \Xi) = 1, \quad \det \Xi \neq 0,$$

let $\Theta_\Xi(N) = \Xi N^T \Xi^\dagger$ and let L_Ξ be the real matrix of this map in Pauli coordinates. Then

$$\begin{aligned} P &= 2\mathbf{E}_A^T L_\Xi \mathbf{E}_B, \\ g &= \mathbf{E}_A^T J \mathbf{E}_A, \quad h = \mathbf{E}_B^T J \mathbf{E}_B, \\ L_\Xi^T J L_\Xi &= |\det \Xi|^2 J, \\ P^T g^{-1} P &= 4|\det \Xi|^2 h. \end{aligned}$$

Proof. If e_i and f_j are the Pauli-coordinate columns of the two frames, the Born rule and the trace pairing give

$$P_{ij} = \text{Tr}(E_i \Theta_\Xi(N_j)) = 2e_i^T L_\Xi f_j,$$

which is the first identity; the two Gram identities are definitions. Moreover,

$$\det \Theta_\Xi(N) = \det(\Xi N^T \Xi^\dagger) = |\det \Xi|^2 \det N.$$

In Pauli coordinates this says $(L_\Xi x)^T J (L_\Xi x) = |\det \Xi|^2 x^T J x$ for every real x . Polarizing the two quadratic forms yields $L_\Xi^T J L_\Xi = |\det \Xi|^2 J$. Finally,

$$\begin{aligned} P^T g^{-1} P &= 4\mathbf{E}_B^T L_\Xi^T \mathbf{E}_A (\mathbf{E}_A^T J \mathbf{E}_A)^{-1} \mathbf{E}_A^T L_\Xi \mathbf{E}_B \\ &= 4\mathbf{E}_B^T L_\Xi^T J L_\Xi \mathbf{E}_B = 4|\det \Xi|^2 h, \end{aligned}$$

because $\mathbf{E}_A (\mathbf{E}_A^T J \mathbf{E}_A)^{-1} \mathbf{E}_A^T = J$. □

The complete table is

$$p_{ij} = r_i^T P r_j, \quad u^T P u = 1.$$

The final identity in lemma 5.2 and $\det \Xi \neq 0$ show that P is invertible. With $Q = g^{-1}$, nullness of Bob's five effects becomes

$$F_j(P, g) := (P r_j)^T Q (P r_j) = 0, \quad j = 1, \dots, 5. \quad (4)$$

6 Physical exactness and smoothness

Theorem 6.1 (Local physical completeness). *Near every genuine residual point, every nearby (P, g) satisfying (3), the five incidence equations (4), normalization, Lorentz signature, and the strict time-orientation inequalities is realized by a genuine residual two-qubit strategy. Every incidence tangent therefore integrates to a two-sided physical curve.*

Proof. Because g has signature $(1, 3)$, $u^T g u = 1$, and the restriction to $u^\perp$ is negative definite, choose at the reference metric three vectors spanning $u^\perp$. For nearby metrics, project these fixed vectors by

$$v \longmapsto v - (v^T g u) u.$$

They remain independent. Ordinary Gram–Schmidt for the positive inner product $-g$ on $u^\perp$ produces a smooth negative orthonormal frame: all denominators stay nonzero in a neighborhood of the reference point. Mapping u to the standard time vector and this frame to the three standard spatial vectors gives a smoothly varying E_A with

$$E_A^T J E_A = g, \quad E_A u = (1, 0, 0, 0)^T.$$

Each r_i is g -null and has positive product with u ; hence $E_A r_i$ is future null and represents a positive rank-one Alice effect. The circuit makes both measurements sum to the identity.

Put $x_j = Pr_j$ and

$$s_j = \frac{1}{2} E_A^{-T} x_j.$$

The factor $1/2$ is forced by the Pauli trace pairing: if S_j has coordinate vector s_j , then the four basis probabilities obey

$$x_j = (\text{Tr}(E_i S_j))_{i=1}^4 = 2 E_A^T s_j.$$

The incidence equations give $s_j^T J s_j = 0$, while $2(s_j)_0 = u^T x_j$ is the strictly positive Bob marginal. Thus s_j represents a future-null positive rank-one steered operator S_j . The circuit yields

$$S_1 + S_2 = S_3 + S_4 + S_5 =: \rho_A.$$

Normalization is explicit:

$$\text{Tr } \rho_A = \text{Tr}(S_1 + S_2) = u^T P u = 1.$$

Full rank is open near the original point. Let $\Xi = \rho_A^{1/2}$ and set

$$N_j^T = \Xi^{-1} S_j \Xi^{-\dagger}.$$

The N_j are positive rank-one effects and normalize on both inputs. The reconstructed pure state

$$|\psi_\Xi\rangle = \sum_{m,n} \Xi_{mn} |mn\rangle$$

is normalized because $\langle \psi_\Xi | \psi_\Xi \rangle = \text{Tr}(\Xi^\dagger \Xi) = \text{Tr } \rho_A = 1$. Moreover

$$\text{Tr}(E_i S_j) = r_i^T P r_j,$$

so every reconstructed probability is the trace of a product of positive operators and is nonnegative, including when that trace is zero.

The five incidence constraints and normalization have independent differentials by the calculation in proposition 6.2. The elementary regular-level-set construction is obtained by choosing six coordinates in which the constraint Jacobian is invertible and solving those coordinates as smooth functions of the remaining fourteen. In that local chart, every kernel vector of the constraint differential is the derivative of a straight coordinate line, hence integrates to a curve in the incidence set. Signature, time orientation, positive marginals, and full rank are open conditions, giving a two-sided physical curve for sufficiently small parameter values even when a joint probability is zero. $\square$

Proposition 6.2 (Smooth incidence manifold). *Restricted to the open genuine residual stratum—in particular, where g has Lorentz signature, all strict time-orientation inequalities hold, and P is invertible—the normalized incidence space is a smooth real 14-dimensional manifold.*

Proof.

$$\nabla_P F_j = 2QPr_j r_j^T.$$

If a linear combination vanishes, invertibility of QP gives

$$\sum_j \mu_j r_j r_j^T = \text{diag}(\mu_1, \dots, \mu_4) + \mu_5 r_5 r_5^T = 0.$$

Off-diagonal entries first give $\mu_5 = 0$, then diagonal entries give all other coefficients zero. Normalization is independent because the radial variation $\delta P = P$ annihilates every dF_j but changes $u^T P u$. There are $16 + 4 - 5 - 1 = 14$ dimensions. $\square$

Remark 6.3. The proposition is local to the open stratum used by the residual equality proof; singular normalized incidence points outside that stratum are not asserted to be smooth. The projected behavior image has dimension at most 14; no claim of exact projected dimension is needed for the equality theorem.

7 Stationarity and strictly positive multipliers

Every entry of the complete residual behavior is $r_i^T P r_j$. Thus, after collecting the original Bell coefficients into numbers γ_{ij} , its restriction to the incidence coordinates is the linear objective

$$\mathcal{L}_c(P) := \sum_{i,j} \gamma_{ij} r_i^T P r_j = \text{Tr}(B_c^T P), \quad B_c := \sum_{i,j} \gamma_{ij} r_i r_j^T.$$

At a stationary point, the six constraint gradients are linearly independent. Thus there are unique multipliers $\alpha, \lambda_1, \dots, \lambda_5$, with sign convention

$$B_c - \alpha u u^T = -2QP\Lambda, \quad \Lambda = \sum_{j=1}^5 \lambda_j r_j r_j^T. \quad (5)$$

Stationarity in the four metric variables says $\lambda \in \ker \mathcal{D}^T$, where $\mathcal{D}$ is defined in section 8.

Lemma 7.1 (Finite POVM duality). *For*

$$\max_{N_j \geq 0, \sum_j N_j = \mathbb{I}} \sum_j \text{Tr}(K_j N_j),$$

there is a Hermitian Γ such that

$$\Gamma \geq K_j, \quad (\Gamma - K_j)N_j = 0.$$

Proof. Weak duality is immediate. Strong duality follows by separating the closed finite-dimensional cone

$$\left\{ \left(\sum_j X_j, \sum_j \text{Tr}(K_j X_j) - t \right) : X_j \geq 0, t \geq 0 \right\}$$

from $(\mathbb{I}, V + \varepsilon)$, where V is the primal optimum, and taking $\varepsilon \downarrow 0$. Equality of primal and dual values gives

$$\sum_j \text{Tr}[(\Gamma - K_j)N_j] = 0.$$

Every summand is nonnegative, hence zero; for positive matrices, $\text{Tr}(AB) = 0$ implies $AB = 0$. Full details are included in appendix E. $\square$

Lemma 7.2 (Determinant pullback for one input). *At a stationary residual strategy, fix one Bob input y , write its effects as $N_{j|y}$, and let $K_{j|y}$ be their Bell score operators. There is a Hermitian normalization multiplier $\widehat{\Gamma}_y$ such that*

$$K_{j|y} - \widehat{\Gamma}_y + 4|\det \Xi|^2 \lambda_{j|y} \text{adj}(N_{j|y}) = 0 \quad (6)$$

for every outcome j . Here, for $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\text{adj}(N) := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

is the matrix adjugate.

Proof. By lemma 5.2,

$$F_j = (Pr_j)^T g^{-1}(Pr_j) = 4|\det \Xi|^2 r_j^T h r_j = 4|\det \Xi|^2 \det N_{j|y}.$$

For an arbitrary Hermitian variation ΔN ,

$$d(\det)_N[\Delta N] = \text{Tr}(\text{adj}(N)\Delta N),$$

as follows by differentiating $ad - bc$. Hold the state, Alice's measurements, and Bob's other input fixed. The first variation of the Bell objective is $\sum_j \text{Tr}(K_{j|y}\Delta N_{j|y})$. Pulling the incidence Lagrangian (5) back to these effect variables, and imposing $\sum_j \Delta N_{j|y} = 0$ with a separate Hermitian multiplier $\widehat{\Gamma}_y$, gives

$$\sum_j \text{Tr} \left[\left(K_{j|y} - \widehat{\Gamma}_y + 4|\det \Xi|^2 \lambda_{j|y} \text{adj}(N_{j|y}) \right) \Delta N_{j|y} \right] = 0.$$

The unconstrained outcome variations are arbitrary after introduction of the normalization multiplier, so every coefficient vanishes, proving (6). $\square$

Proposition 7.3 (Strict multiplier positivity). *At a residual global Bell maximizer whose value is strictly above the PVM support value,*

$$\lambda_j > 0 \quad (j = 1, \dots, 5), \quad \Lambda > 0.$$

Proof. For each Bob input y , hold the state and all other measurements fixed and collect the terms of $\mathcal{L}_c$ depending on that input as

$$\sum_{j \in B_y} \text{Tr}(K_{j|y} N_{j|y}).$$

Apply lemma 7.1 separately to Bob's binary and ternary inputs, with distinct dual operators Γ_y . Complementary slackness says $(\Gamma_y - K_{j|y})N_{j|y} = 0$. Since $N_{j|y}$ is nonzero rank one, every positive semidefinite matrix annihilating its range is a nonnegative multiple of its adjugate. Thus

$$\Gamma_y - K_{j|y} = \tau_{j|y} \text{adj}(N_{j|y}), \quad \tau_{j|y} \geq 0,$$

$$\text{adj}(N_{j|y})N_{j|y} = 0.$$

It remains to identify the normalization multiplier $\widehat{\Gamma}_y$ from lemma 7.2 with the finite-POVM dual Γ_y . Both differences from $K_{j|y}$ annihilate $\text{ran } N_{j|y}$, so $(\widehat{\Gamma}_y - \Gamma_y)N_{j|y} = 0$ for every j . Those

542 ranges span $\mathbb{C}^2$, because $\sum_j N_{j|y} = \mathbb{I}$; hence $\hat{\Gamma}_y = \Gamma_y$. Comparing the two adjugate coefficients
 543 gives

$$544 \quad 4|\det \Xi|^2 \lambda_{j|y} = \tau_{j|y} \geq 0.$$

545 If $\lambda_{j|y} = 0$, then $K_{j|y} = \Gamma_y$. Replacing that entire input by the deterministic PVM $N_{j|y} = \mathbb{I}$,
 546 $N_{k|y} = 0$ for $k \neq j$, preserves the score because both the old optimum and the replacement score
 547 equal $\text{Tr } \Gamma_y$.

548 For completeness, the resulting behavior has an explicit local model. Let y_* be Bob's remaining
 549 nontrivial input, let $q(\ell) = p_B(\ell|y_*)$, and omit zero-weight ℓ . Use hidden variable ℓ , Alice response

$$550 \quad p_A(a|x, \ell) = \frac{p(a, \ell|x, y_*)}{q(\ell)},$$

551 and Bob response ℓ on y_* and the fixed label j on the replaced input. This reproduces the y_* block
 552 by construction and the deterministic block because nonsignaling gives $\sum_\ell p(a, \ell|x, y_*) = p_A(a|x)$.
 553 Expanding the local response probabilities into deterministic choices yields a convex combination
 554 of identity-and-zero qubit PVM behaviors. A strict value above the PVM support therefore forces
 555 every $\tau_{j|y}$, and hence every λ_j , to be positive. Since $r_1, \dots, r_4$ form a basis, $\Lambda > 0$. $\square$

556 8 Exact second variation

557 A tangent metric variation has the form

$$558 \quad H(\dot{a}, \dot{b}, \dot{c}, \dot{d}) = \begin{pmatrix} 0 & 0 & \dot{a} & \dot{b} \\ 0 & 0 & \dot{c} & \dot{d} \\ \dot{a} & \dot{c} & 0 & \dot{a} + \dot{b} + \dot{c} + \dot{d} \\ \dot{b} & \dot{d} & \dot{a} + \dot{b} + \dot{c} + \dot{d} & 0 \end{pmatrix}.$$

559 Define

$$560 \quad \Phi(y) = \begin{pmatrix} y_2(y_0 + y_3) \\ y_3(y_0 + y_2) \\ y_2(y_1 + y_3) \\ y_3(y_1 + y_2) \end{pmatrix}, \quad y_j = QPr_j,$$

561 and

$$562 \quad \mathcal{D} : H \mapsto (y_1^T H y_1, \dots, y_5^T H y_5).$$

563 Then

$$564 \quad \text{rank } \mathcal{D} = \dim \text{span}\{\Phi(y_1), \dots, \Phi(y_5)\}.$$

565 Let $K = \ker \mathcal{D}^T$, $k = \dim K = 5 - \text{rank } \mathcal{D}$. Metric stationarity says $\lambda \in K$.

566 For a tangent $(\delta P, H)$, set

$$567 \quad S = P^T Q P, \quad \delta P = P W + H Q P.$$

568 Since $dQ = -Q H Q$, direct differentiation gives

$$569 \quad \begin{aligned} dF_j &= 2(QPr_j)^T \delta P r_j - (QPr_j)^T H (QPr_j) \\ &= 2r_j^T S W r_j + y_j^T H y_j. \end{aligned}$$

570 Thus the first-order equations $dF_j = 0$ ask the metric differential $\mathcal{D}(H)$ to equal the vector with
 571 entries $-2r_j^T S W r_j$. Finite-dimensional linear algebra says this equation is solvable exactly when

its right side is orthogonal to $K = \ker \mathcal{D}^T$. Indeed, $\text{ran } \mathcal{D} = (\ker \mathcal{D}^T)^\perp$, because both spaces have dimension $\text{rank } \mathcal{D}$. Therefore the Fredholm compatibility conditions are

$$\text{Tr}(SW\Lambda_\mu) = 0 \quad (\mu \in K), \quad \Lambda_\mu = \sum_j \mu_j r_j r_j^T.$$

The compatible space

$$\mathcal{H}_K = \{W : \text{Tr}(SW\Lambda_\mu) = 0 \ \forall \mu \in K\}$$

has dimension $16 - k$. To see that the k displayed functionals are independent, suppose a combination indexed by $\nu \in K$ vanishes for every W . Then $\text{Tr}(SW\Lambda_\nu) = 0$ for all W , so $\Lambda_\nu S = 0$. The matrix $S = P^T Q P$ is invertible, and the map $\nu \mapsto \Lambda_\nu = \sum_j \nu_j r_j r_j^T$ is injective by proposition 6.2; hence $\nu = 0$.

Proposition 8.1 (Weighted second form).

$$\sum_j \lambda_j D^2 F_j = 2q(W), \quad q(W) = \text{Tr}(SW\Lambda W^T),$$

and

$$\text{inertia}(q) = (4, 12).$$

Every positive- q compatible tangent can be adjusted to satisfy normalization without changing q .

Proof. Differentiating g^{-1} twice and completing the square after $\delta P = PW + HQP$ gives

$$D^2 F_j = 2(PW r_j)^T Q (PW r_j),$$

and summing against λ_j yields the trace formula. The exact expansion is recorded in appendix F and independently checked by the verifier.

The matrix S has signature $(1, 3)$, while $\Lambda > 0$. Invertible congruences therefore reduce q to

$$q_0(Z) = \sum_{j=1}^4 Z_{1j}^2 - \sum_{i=2}^4 \sum_{j=1}^4 Z_{ij}^2,$$

giving inertia $(4, 12)$.

For $\mu \in K$,

$$\text{Tr}(S\Lambda_\mu) = \sum_j \mu_j r_j^T S r_j = \sum_j \mu_j F_j = 0.$$

Hence the identity matrix belongs to $\mathcal{H}_K$. Taking $\mu = \lambda$ also gives $q(\mathbb{I}) = \text{Tr}(S\Lambda) = 0$, while compatibility gives the polar cross term $\text{Tr}(SW\Lambda) = 0$. Thus $q(W + t\mathbb{I}) = q(W)$. Adding $t\mathbb{I}$ changes the normalization derivative by

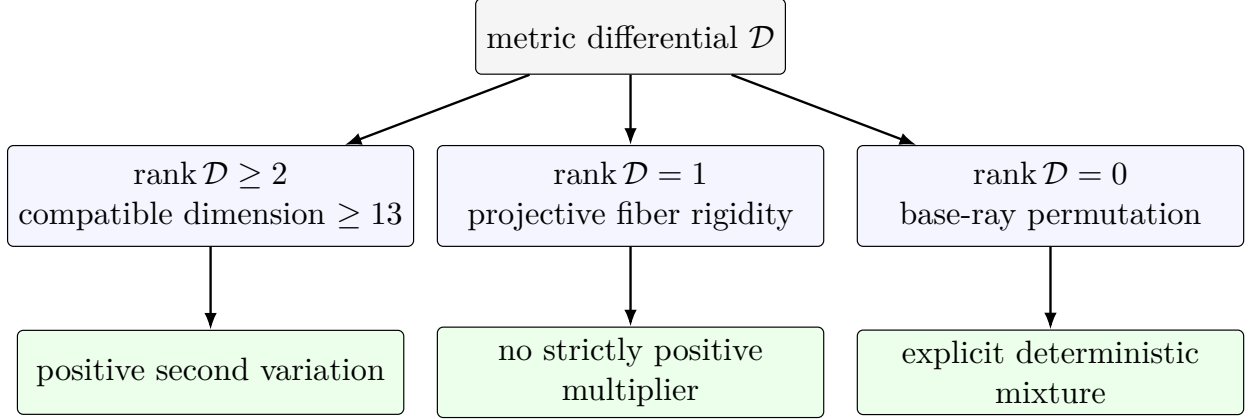
$$t u^T P u = t,$$

so choose t to impose $u^T \delta P u = 0$. □

With the convention (5), a normalized physical curve has

$$\frac{d^2}{dt^2} \mathcal{L}_c(P(t)) \big|_{t=0} = 2q(W).$$

Hence local maximality requires $q \leq 0$ on every normalized tangent.



Every rank contradicts a strict non-PVM Bell maximum.

Figure 3: The exhaustive rank trichotomy for the metric differential.

9 Rank trichotomy and universal equality

9.1 Rank at least two

If $\text{rank } \mathcal{D} \geq 2$, then $k \leq 3$ and

$$\dim \mathcal{H}_K = 16 - k \geq 13.$$

A subspace on which a form of inertia $(4, 12)$ is nonpositive has dimension at most 12: projection to its twelve negative coordinates is injective. Therefore $\mathcal{H}_K$ contains W with $q(W) > 0$. Normalize it using proposition 8.1 and integrate it using theorem 6.1. The resulting physical curve has positive Bell second derivative, contradicting local maximality.

9.2 Rank one

The coefficient form of the metric differential is the quadratic map

$$\Phi(y) = (y_2(y_0 + y_3), y_3(y_0 + y_2), y_2(y_1 + y_3), y_3(y_1 + y_2))^T.$$

Proposition 9.1 (Projective fiber theorem). *On the strict g -null quadric, the base locus of Φ is exactly the five coefficient rays $[r_1], \dots, [r_5]$. Away from the base locus, Φ is projectively injective, including on the exceptional divisors*

$$y_2 = 0, \quad y_3 = 0, \quad y_0 = y_1, \quad y_2 = y_3.$$

Proof. The generic rational inverse and the exact exceptional-divisor calculations are given in appendix G. Each resultant and factorization is reproduced by the exact verifier. The strict metric inequalities exclude extra null roots without excluding any genuine residual point. $\square$

If $\text{rank } \mathcal{D} = 1$, all nonzero $\Phi(y_j)$ are projectively equal. The five $y_j = QPr_j$ are distinct rays because QP is invertible. Proposition 9.1 therefore permits at most one nonzero row of $\mathcal{D}$. Consequently

$$\ker \mathcal{D}^T = \{\mu : \mu_{j_0} = 0\}$$

for some j_0 , which contains no strictly positive vector. This contradicts $\lambda_j > 0$.

9.3 Rank zero

If $\text{rank } \mathcal{D} = 0$, every y_j belongs to the five-ray base locus. Invertibility gives

$$y_j = t_j r_{\pi(j)}, \quad t_j > 0,$$

for a permutation π . Let $\varepsilon = (1, 1, -1, -1, -1)^T$. The 4×5 matrix $[r_1 \cdots r_5]$ has one-dimensional kernel $\mathbb{R}\varepsilon$. Applying the invertible map QP to its circuit gives

$$\sum_j \varepsilon_j t_j r_{\pi(j)} = 0.$$

This coefficient vector, reordered by π , must be a scalar multiple of ε . Positivity of the t_j , together with the unequal cardinalities of the positive and negative parts of ε , forces π to preserve the binary pair $\{1, 2\}$ and ternary triple $\{3, 4, 5\}$; coefficient comparison then forces every t_j to have one common positive value t . After relabeling, $QP = t\mathbb{I}$, hence $P = tg$. Since $u^T Pu = u^T gu = 1$, one has $t = 1$, and therefore

$$QP = \mathbb{I}, \quad P = g.$$

Proposition 9.2 (Rank-zero deterministic simulation). *Every rank-zero probability table $p_{ij} = r_i^T g r_j$ is a convex combination of deterministic local behaviors, and hence belongs to $\overline{\mathcal{Q}}_2^{\text{PVM}}$.*

Proof. For $i, j \in \{3, 4, 5\}$, define directly from the Bell table

$$a_i := p(1, i|0, 1), \quad b_i := p(2, i|0, 1), \quad c_{ij} := p(i, j|1, 1).$$

Because $P = g = g^T$, the other mixed block satisfies

$$p(i, 1|1, 0) = a_i, \quad p(i, 2|1, 0) = b_i.$$

Moreover $c_{ij} = c_{ji}$, and nullness gives $c_{ii} = 0$. Normalization and the two local circuits give

$$\sum_{j \neq i} c_{ij} = a_i + b_i, \quad \sum_i a_i = \sum_i b_i = \frac{1}{2}.$$

Appendix H constructs a matrix q_{ij} satisfying

$$q_{ii} = 0, \quad 0 \leq q_{ij} \leq c_{ij}, \quad \sum_j q_{ij} = b_i, \quad \sum_i q_{ij} = a_j.$$

For each ordered ternary pair (i, j) , use the two deterministic branches

weight	Alice binary	Bob binary	ternary outputs (A, B)
q_{ij}	1	2	(i, j)
$c_{ij} - q_{ij}$	2	1	(i, j)

The weights with ternary outputs (i, j) sum to c_{ij} , reproducing the ternary–ternary block. For fixed Alice ternary output i , the Bob-binary outcome-2 mass is $\sum_j q_{ij} = b_i$, and its outcome-1 mass is $\sum_j (c_{ij} - q_{ij}) = a_i$. For fixed Bob ternary output j , the Alice-binary outcome-1 mass is $\sum_i q_{ij} = a_j$, and its outcome-2 mass is b_j . Finally,

$$\sum_{i,j} q_{ij} = \frac{1}{2}, \quad \sum_{i,j} (c_{ij} - q_{ij}) = \frac{1}{2},$$

so the binary–binary block has probabilities $1/2$ on $(1, 2)$ and $(2, 1)$, and zero on $(1, 1)$ and $(2, 2)$, exactly as $r_k^T g r_\ell$ for $k, \ell \in \{1, 2\}$. The four blocks are therefore reproduced simultaneously. Identity and zero projectors implement every deterministic branch. $\square$

9.4 Equality

Theorem 9.3 (Universal two-input equality). *For every finite input-dependent output architecture $(A_0, A_1; B_0, B_1)$,*

$$\overline{Q}_2^{\text{POVM}}(\mathbf{A}, \mathbf{B}) = \overline{Q}_2^{\text{PVM}}(\mathbf{A}, \mathbf{B}).$$

Proof. The PVM-to-POVM inclusion is immediate. Suppose the reverse inclusion failed. The nearest-point argument in lemma 2.3 would give a Bell functional whose global POVM maximum strictly exceeds its PVM maximum. Choose an extreme point of its exposed maximizing face. By theorem 4.5, a maximizing component lies in the genuine residual architecture. Every local measurement is therefore globally optimal with the other variables fixed, so proposition 7.3 applies.

The metric differential has rank 0, 1, 2, 3, or 4. Rank at least two has an uphill physical direction; rank one cannot carry the required positive multiplier; rank zero is PVM-simulable by proposition 9.2. Every case contradicts a strict value above the PVM support function. $\square$

Corollary 9.4 (Minimum input architecture). *Up to exchanging the parties, 3×2 is the minimum input architecture for a strict fixed-qubit POVM-over-PVM Bell separation.*

Proof. With one input on either party every nonsignaling behavior is local, hence a deterministic PVM behavior maximizes every Bell functional. Two inputs on both parties are excluded by theorem 9.3. Theorem 3.1 supplies an exact 3×2 separation. $\square$

10 Operational consequences and limitations

For every Bell functional with two inputs per party, optimizing over fixed-qubit PVMs already provides the fixed-qubit POVM bound. Thus a value above that bound excludes all local-qubit realizations, not merely projective ones.

Likewise, no qubit behavior with two inputs per party can certify that a nonprojective local measurement was indispensable: after shared-randomness convexification, a shared-randomness mixture of PVM realizations always exists. This does not identify the state or measurements in a particular laboratory realization, and it does not assert raw-set equality without shared randomness.

Equivalently, the theorem rules out every two-input linear Bell-functional separation for bipartite qubits after shared-randomness convexification. It does not prove equality of the raw nonconvex behavior sets.

We do not claim the exact global POVM or PVM optimum of (1). We also do not infer a private-randomness entropy bound: such a statement requires an adversarial extension of the convex decomposition, not only equality of observed behavior sets.

11 Discussion

The threshold result is special to local dimension two. Natural open questions include minimum-input laws in dimension three and higher, multipartite scenarios, equality without convexification, constructive simulation algorithms for generic two-input behaviors, instruments and sequential measurements, and experimentally robust exact 3×2 witnesses. Determining the exact two-qubit POVM and PVM optima of (1) is also open but unnecessary for the separation.

The proof illustrates a distinction between operator-level and behavioral generality. Ternary qubit POVMs are genuine generalized measurements, yet two inputs do not provide enough Bell structure to make them indispensable after the state and shared-randomness decomposition may

change. A third input can lock the state and steering geometry while reserving another measurement for a discrimination advantage.

Data, code, and verification

The accompanying repository contains exact Bell coefficients and strategies, the separation verifier, the exceptional-fiber identities, the closure verifier, and an independent rank-zero simulator. One offline command runs all exact checks. The code verifies finite algebraic identities and explicit constructions; the convexity, extremality, physical-reconstruction, and dimension arguments remain explicit manuscript proofs.

AI-assisted research and verification

Generative-AI systems were used substantively throughout the research workflow, including mathematical exploration and derivation, adversarial proof analysis, symbolic and software work, literature support, and manuscript preparation. The work was developed through an iterative, human-directed research process. The author determined the scope and claims released here, curated the resulting package, and assumes responsibility for the submitted manuscript and public verification materials. The proof and exact artifacts are supplied to make the AI-mediated research independently checkable; no claim of prior independent expert human verification is made.

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A Exact algebra for the 3×2 separation

We record the identities that make the projective bound independent of numerical optimization. For a pair involving auxiliary label 2, let

$$K_{\mathbf{n}} = \frac{3}{5}D\Pi(\mathbf{n})D, \quad K_{\mathbf{m}} = \frac{4}{5}D\Pi(\mathbf{m})D,$$

where $x = n_z$, $y = m_z$, $r = \mathbf{n} \cdot \mathbf{m}$, and

$$D = \text{diag}\left(\sqrt{\frac{1+\eta}{2}}, \sqrt{\frac{1-\eta}{2}}\right).$$

Direct expansion, using $\|\mathbf{n}\| = \|\mathbf{m}\| = 1$, gives

$$\det(K_{\mathbf{n}} - K_{\mathbf{m}}) = -\frac{3}{50}(1 - \eta^2)(1 - r) \leq 0.$$

Thus $K_{\mathbf{n}} - K_{\mathbf{m}}$ has eigenvalues of opposite signs, including a possible zero boundary eigenvalue, and the optimized complementary-projector score is

$$F_{\eta} = \frac{\text{Tr } K_{\mathbf{n}} + \text{Tr } K_{\mathbf{m}} + \|K_{\mathbf{n}} - K_{\mathbf{m}}\|_1}{2}.$$

726 Its discriminant yields

$$727 \quad 20F_\eta = 7 + \eta(3x + 4y) + \sqrt{[-1 + \eta(3x - 4y)]^2 + 24(1 - \eta^2)(1 - r)}.$$

728 At $\eta = 0$,

$$729 \quad F_0(r) = \frac{7}{20} + \frac{1}{20}\sqrt{25 - 24r}.$$

730 Set

$$731 \quad q = \sqrt{25 - 24r} \geq 1, \quad H = 8 - 3x - 4y \geq 1.$$

732 To prove $F_\eta \leq F_0(r) + 2\eta/5$, it is enough to prove

$$733 \quad \sqrt{[-1 + \eta(3x - 4y)]^2 + 24(1 - \eta^2)(1 - r)} \leq q + \eta H.$$

734 The right side is at least one, so squaring is valid. Using $q^2 = 25 - 24r$, the squared difference is
735 exactly

$$736 \quad \begin{aligned} & 2\eta\{\eta[8(1 - y)(4 - 3x) + 12(1 - r)] \\ & + (q - 1)(8 - 3x - 4y) + 8(1 - y)\}. \end{aligned}$$

737 Every factor is nonnegative on

$$738 \quad 0 \leq \eta \leq 1, \quad -1 \leq x, y, r \leq 1.$$

739 No omitted compatibility condition among x, y, r can invalidate the estimate because it holds on
740 the larger box.

741 For $u = |\mathbf{b}_0 \cdot \mathbf{b}_1|$, both relevant pairs satisfy $r \geq -u$. Moreover,

$$742 \quad (5 + 8u)^2 - (25 + 24u) = 8u(7 + 8u) \geq 0,$$

743 so

$$744 \quad F_0(r) \leq \frac{7}{20} + \frac{1}{20}\sqrt{25 + 24u} \leq \frac{3}{5} + \frac{2u}{5}.$$

745 For completeness, the scalar inequalities in the CHSH-deficit estimate have nonnegative squared
746 differences

$$747 \quad \left(1 - \frac{u^2}{2}\right)^2 - (1 - u^2) = \frac{u^4}{4},$$

$$748 \quad \left(2 - \frac{u^2}{4}\right)^2 - (4 - u^2) = \frac{u^4}{16}.$$

750 Their right sides are nonnegative on $0 \leq u \leq 1$, so no sign reversal is introduced by squaring.

751 B The strengthened algebraic strategy

752 **Proposition B.1** (Strengthened attained lower bound). *There is a fixed-qubit POVM strategy*
753 *attaining*

$$754 \quad L_1 = \frac{16 + 8\sqrt{7813}}{25} > U.$$

755 Thus L_1 is an attained POVM lower bound. No global POVM optimality is asserted.

Proof. Put

$$q = \sqrt{7813}, \quad \eta = \frac{1}{q}, \quad s = \sqrt{1 - \eta^2} = \frac{6\sqrt{217}}{q}.$$

Use

$$|\psi_\eta\rangle = \sqrt{\frac{1+\eta}{2}}|00\rangle + \sqrt{\frac{1-\eta}{2}}|11\rangle,$$

Bob observables $B_0 = X, B_1 = Z$, and Alice observables

$$A_0 = \frac{6\sqrt{217}X + qZ}{125}, \quad A_1 = \frac{6\sqrt{217}X - qZ}{125}.$$

They square to the identity because

$$36 \cdot 217 + 7813 = 125^2.$$

The CHSH value is $S = 250/q$.

Set

$$k = \frac{3}{5}\sqrt{\frac{q-1}{q+1}}$$

and

$$M_0 = \frac{1}{2} \begin{pmatrix} k^2 & k \\ k & 1 \end{pmatrix}, \quad M_1 = \frac{1}{2} \begin{pmatrix} k^2 & -k \\ -k & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 - k^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

These are positive rank-one effects and sum to the identity. The diagonal dual operator

$$\Gamma = \text{diag}\left(\frac{2}{5}(1 + \eta), \frac{6}{25}(1 - \eta)\right)$$

dominates all three weighted steered score operators and annihilates the corresponding effect ranges.

Hence

$$T = \text{Tr } \Gamma = \frac{16 + 4\eta}{25}, \quad 10S + T = \frac{16 + 8q}{25}.$$

For the strict comparison with U ,

$$(6 + 80\sqrt{7813})^2 - 2 \cdot 5003^2 = -56782 + 960\sqrt{7813},$$

and

$$960^2 \cdot 7813 - 56782^2 = 3\,976\,265\,276 > 0.$$

Both sides being compared are positive, these two exact square comparisons prove $L_1 > U$. $\square$

Remark B.2 (Scope of the strengthening). Within the one-parameter family used above,

$$f(\eta) = 20\sqrt{2 - \eta^2} + \frac{16 + 4\eta}{25}$$

has

$$f'(\eta) = -\frac{20\eta}{\sqrt{2 - \eta^2}} + \frac{4}{25}, \quad f''(\eta) = -\frac{40}{(2 - \eta^2)^{3/2}} < 0.$$

Its unique critical point is $\eta = 1/\sqrt{7813}$. This proves optimality only within that one-parameter family and is not a global POVM optimality claim.

C Degenerate cases in the one-binary-party theorem

If B_0, B_1 are nondegenerate and have linearly independent Bloch axes, Φ_B maps the positive cone onto a three-dimensional Lorentz cone. An extreme ray of the image lies on the boundary ellipse. Its minimum-norm Bloch lift in the plane of the two axes has norm one; a perpendicular component would exceed the Bloch ball. Thus the positive lift in $\text{span}_{\mathbb{R}}\{\mathbb{I}, B_0, B_1\}$ is unique and rank one.

If the Bloch axes are parallel or antiparallel, the image is the cone over an interval and has two extreme rays. Every positive relation reduces to one-versus-one cancellations or a two-versus-two relation with repeated zero-padded outcomes. The same purification construction applies.

If one observable is scalar and the other nondegenerate, the image is again two-dimensional; the scalar input is deterministic and does not add a coordinate. If both are scalar, Bob is deterministic on both inputs and the behavior is local. A rank-deficient circuit sum Ω forces all lifted rays to be collinear and is realized by a product purification. Zero effects receive zero projectors. These cases also show that circuit weights remain nonnegative and sum to one at the boundary.

D Extremal qubit POVMs and the rank-square bound

Let M_a be the nonzero effects of a POVM. For each a , let $\mathcal{V}_a$ be the real vector space of Hermitian operators supported on $\text{ran } M_a$. It has dimension $\text{rank}(M_a)^2$. Consider the summation map

$$\bigoplus_a \mathcal{V}_a \longrightarrow \text{Herm}(2), \quad (H_a)_a \longmapsto \sum_a H_a.$$

If it has a nonzero kernel, then for sufficiently small $\varepsilon > 0$, $M_a \pm \varepsilon H_a \geq 0$, and the two perturbed tuples are distinct POVMs whose average is M . Extremality therefore requires injectivity, giving

$$\sum_a \text{rank}(M_a)^2 \leq \dim_{\mathbb{R}} \text{Herm}(2) = 4.$$

This necessary condition is all that is used in theorem 4.5. Together with linear independence in a three-dimensional span, it leaves exactly three rank-one effects.

E Finite-dimensional POVM duality

Let

$$V = \max_{\substack{N_j \geq 0 \\ \sum_j N_j = \mathbb{I}}} \sum_j \text{Tr}(K_j N_j).$$

Weak duality says that every $\Gamma \geq K_j$ obeys

$$\sum_j \text{Tr}(K_j N_j) \leq \text{Tr } \Gamma.$$

For the reverse direction, consider the closed cone

$$\mathcal{C} = \left\{ \left(\sum_j X_j, \sum_j \text{Tr}(K_j X_j) - t \right) : X_j \geq 0, t \geq 0 \right\} \subset \text{Herm}(2) \times \mathbb{R}.$$

Closedness follows because convergence of $\sum_j X_j$ bounds every $0 \leq X_j \leq \sum_k X_k$.

The point $(\mathbb{I}, V + \varepsilon)$ is outside $\mathcal{C}$. Nearest-point separation provides a nonzero functional

$$\ell(A, s) = \text{Tr}(\Gamma_\varepsilon A) + \gamma s$$

nonnegative on $\mathcal{C}$ and negative at that point. Since $(0, -t) \in \mathcal{C}$, $\gamma \leq 0$. If $\gamma = 0$, positivity on all $(X, \text{Tr}(K_j X))$ would give $\Gamma_\varepsilon \geq 0$, contradicting $\text{Tr} \Gamma_\varepsilon < 0$. Normalize $\gamma = -1$. Taking one nonzero $X_j = X$ gives $\Gamma_\varepsilon \geq K_j$, while separation and weak duality give

$$V \leq \text{Tr} \Gamma_\varepsilon < V + \varepsilon.$$

Fix one index j_0 . Since $\Gamma_\varepsilon - K_{j_0} \geq 0$ and $\text{Tr} \Gamma_\varepsilon < V + \varepsilon$, the positive matrix $\Gamma_\varepsilon - K_{j_0}$ has uniformly bounded trace and hence uniformly bounded operator norm. Thus Γ_ε is bounded. A convergent subsequence as $\varepsilon \downarrow 0$ yields dual attainment with $\text{Tr} \Gamma = V$.

At a primal optimum,

$$0 = \text{Tr} \Gamma - \sum_j \text{Tr}(K_j N_j) = \sum_j \text{Tr}[(\Gamma - K_j) N_j].$$

Each term is nonnegative, hence zero. For $A, B \geq 0$, $\text{Tr}(AB) = 0$ implies $A^{1/2} B A^{1/2} = 0$, and therefore $AB = 0$. This proves complementary slackness.

F Square completion for the incidence Hessian

Let $x_j = P r_j$, $Q = g^{-1}$, and vary

$$P(t) = P + t\delta P, \quad g(t) = g + tH$$

to first order. The inverse expansion is

$$(g + tH)^{-1} = Q - tQHQ + t^2QHQHQ + O(t^3).$$

Substitution in $F_j = (P r_j)^T Q (P r_j)$ gives

$$\begin{aligned} \frac{1}{2} D^2 F_j &= (\delta P r_j)^T Q (\delta P r_j) - 2(\delta P r_j)^T Q H Q (P r_j) \\ &\quad + (P r_j)^T Q H Q H Q (P r_j). \end{aligned}$$

Writing

$$\delta P = P W + H Q P$$

turns the right side into

$$(P W r_j)^T Q (P W r_j).$$

Therefore

$$\sum_j \lambda_j D^2 F_j = 2 \sum_j \lambda_j r_j^T W^T P^T Q P W r_j = 2 \text{Tr}(S W \Lambda W^T).$$

This expansion also fixes the sign convention used at a Bell maximum.

G Projective fibers of the quadratic null-ray map

Let

$$\mathcal{N}_g = \{[x] \in \mathbb{RP}^3 : x^T g x = 0\}$$

for a strict metric (3). To avoid collisions with the state and metric notation, write

$$\Phi(x) = (\xi_0(x), \xi_1(x), \xi_2(x), \xi_3(x)),$$

where

$$\begin{aligned} \xi_0(x) &= x_2(x_0 + x_3), & \xi_1(x) &= x_3(x_0 + x_2), \\ \xi_2(x) &= x_2(x_1 + x_3), & \xi_3(x) &= x_3(x_1 + x_2). \end{aligned}$$

For a fixed target we suppress the argument x .

G.1 Base locus and generic inverse

If $x_2 = x_3 = 0$, nullness gives $x_0 x_1 = 0$, hence r_1, r_2 . If exactly one of x_2, x_3 vanishes, the remaining equations give r_3 or r_4 . If $x_2 x_3 \neq 0$, all four components of Φ vanish only when

$$x_0 = x_1 = -x_2 = -x_3,$$

the ray r_5 . Thus the base locus consists exactly of the five coefficient rays.

The identities

$$\begin{aligned} \xi_0 - \xi_1 &= x_0(x_2 - x_3), & \xi_2 - \xi_3 &= x_1(x_2 - x_3), \\ \xi_0 - \xi_2 &= x_2(x_0 - x_1), & \xi_1 - \xi_3 &= x_3(x_0 - x_1), \\ \xi_0 \xi_3 - \xi_1 \xi_2 &= x_2 x_3 (x_0 - x_1)(x_2 - x_3) \end{aligned}$$

give

$$\begin{pmatrix} (\xi_0 - \xi_1)(\xi_0 - \xi_2)(\xi_1 - \xi_3) \\ (\xi_2 - \xi_3)(\xi_0 - \xi_2)(\xi_1 - \xi_3) \\ (\xi_0 \xi_3 - \xi_1 \xi_2)(\xi_0 - \xi_2) \\ (\xi_0 \xi_3 - \xi_1 \xi_2)(\xi_1 - \xi_3) \end{pmatrix} = \omega(x)x,$$

where

$$\omega(x) = x_2 x_3 (x_2 - x_3)(x_0 - x_1)^2.$$

Hence $[x]$ is recovered from $[\Phi(x)]$ whenever $\omega(x) \neq 0$.

G.2 Exceptional divisor $x_2 = 0$

Fix a nonbase source with $x_2 = 0$. The base-locus analysis shows that $x_3 \neq 0$, and its nonzero target has the form $[0 : \xi_1 : 0 : \xi_3]$. For a candidate preimage z , the two target zeros give

$$z_2(z_0 + z_3) = z_2(z_1 + z_3) = 0.$$

On the direct branch $z_2 = 0$, a nonzero target forces $z_3 \neq 0$. Scale $z_3 = 1$. Equality of target ratios then gives $(z_0, z_1) = t(\xi_1, \xi_3)$, and nullness is

$$t^2 \xi_1 \xi_3 + 2t(b\xi_1 + d\xi_3) = 0.$$

The root $t = 0$ is r_4 ; after removing it, the remaining equation is linear in t . It cannot vanish identically: if $\xi_1 \xi_3 = 0$, positivity of b, d and $(\xi_1, \xi_3) \neq (0, 0)$ gives $b\xi_1 + d\xi_3 \neq 0$. Hence there is at most one nonbase direct preimage.

On the cross branch $z_2 \neq 0$, the target zeros force $z_0 = z_1 = -z_3$. A nonzero target also forces $z_3 \neq 0$, so scale $z_3 = 1$ and write $z_2 = t$. The two nonzero target coordinates of this branch are equal, so a cross preimage can occur only if $\xi_1 = \xi_3$. Its null equation is

$$z^T g z = (1 - 2(b + d))(1 - t).$$

Strictness $b + d < 1/2$ gives only $t = 1$, for which $z = -r_5$ and $\Phi(z) = 0$. Thus the cross branch contributes no nonbase preimage.

G.3 Exceptional divisor $x_3 = 0$

Now fix a nonbase source with $x_3 = 0$. Then $x_2 \neq 0$ and the target is $[\xi_0 : 0 : \xi_2 : 0]$. The target zeros give either the direct branch $z_3 = 0$, or, when $z_3 \neq 0$, the cross equations $z_0 = z_1 = -z_2$.

On the direct branch, a nonzero target permits the scale $z_2 = 1$, and $(z_0, z_1) = t(\xi_0, \xi_2)$. Nullness is

$$t^2 \xi_0 \xi_2 + 2t(a\xi_0 + c\xi_2) = 0.$$

As above, $t = 0$ is the base ray r_3 , and positivity of a, c excludes an identically zero residual linear equation. Thus there is at most one nonbase direct preimage.

On the cross branch, scale $z_2 = 1$ and write $z_3 = t$. Such a target must satisfy $\xi_0 = \xi_2$, and nullness becomes

$$(1 - 2(a + c))(1 - t) = 0.$$

Strictness $a + c < 1/2$ again leaves only $t = 1$, the base ray $-r_5$. Thus this cross branch also contributes no nonbase preimage.

G.4 Exceptional divisor $x_0 = x_1$

If $x_0 = x_1 = 0$, nullness reduces to $2ex_2x_3 = 0$. Since $e > 0$, the point is r_3 or r_4 , hence in the base locus. A nonbase point may therefore be scaled to $x_0 = x_1 = 1$. Put $p = x_2, q = x_3$. The null and target-ratio equations are

$$N_{01} = 1 + 2(a + c)p + 2(b + d)q + 2epq = 0,$$

$$R_{01} = \xi_1 p(1 + q) - \xi_0 q(1 + p) = 0.$$

Here the target is $[\xi_0 : \xi_1 : \xi_0 : \xi_1]$ and is nonzero. Conversely, equality of the first and third, and of the second and fourth, target coordinates for a candidate z gives

$$z_2(z_0 - z_1) = z_3(z_0 - z_1) = 0.$$

If $z_0 \neq z_1$, then $z_2 = z_3 = 0$, so nullness makes z a base ray and $\Phi(z) = 0$. Thus every nonbase candidate is on the direct branch $z_0 = z_1$, and the preceding zero-coordinate check justifies the scale $z_0 = z_1 = 1$.

The exact resultant in p is

$$\text{Res}_p(N_{01}, R_{01}) = -(q + 1) [\xi_1 + q(\xi_0(2a + 2c - 1) + 2\xi_1(b + d))].$$

At $q = -1$,

$$N_{01} = (1 - 2(b + d))(1 + p),$$

so strictness gives $p = -1$, the base ray r_5 . The second resultant factor is not identically zero: if $\xi_1 \neq 0$, its constant term is nonzero; if $\xi_1 = 0$, then $\xi_0 \neq 0$ and $2(a + c) - 1 \neq 0$. It therefore supplies at most one nonbase value of q .

For fixed q , the null equation is linear in p , with coefficient $2(a + c) + 2eq$ and constant term $1 + 2(b + d)q$. Both could vanish only if

$$2(a + c)(b + d) = a + b + c + d - \frac{1}{2},$$

equivalent to

$$(2(a + c) - 1)(2(b + d) - 1) = 0,$$

which strictness excludes. Thus N_{01} determines at most one p for the remaining q , proving uniqueness on this divisor.

G.5 Exceptional divisor $x_2 = x_3$

If $x_2 = x_3 = 0$, nullness gives $x_0x_1 = 0$, hence the base rays r_1, r_2 . For a nonbase point scale $x_2 = x_3 = 1$ and put $p = x_0, q = x_1$. The equations are

$$N_{23} = pq + 2(a + b)p + 2(c + d)q + 2e = 0,$$

$$R_{23} = \xi_2(p + 1) - \xi_0(q + 1) = 0.$$

The target is $[\xi_0 : \xi_0 : \xi_2 : \xi_2]$ and is nonzero. A candidate with these equal coordinate pairs obeys

$$z_0(z_2 - z_3) = z_1(z_2 - z_3) = 0.$$

The direct branch is $z_2 = z_3$, where the base-locus check above justifies the scale $z_2 = z_3 = 1$. On the cross branch $z_2 \neq z_3$, one has $z_0 = z_1 = 0$; nullness becomes $2ez_2z_3 = 0$. Since $e > 0$, the candidate is r_3 or r_4 and has zero target. Hence no nonbase cross preimage exists.

On the direct branch the exact resultant is

$$\text{Res}_p(N_{23}, R_{23}) = -(q + 1) [\xi_0q + 2\xi_0(a + b) + 2\xi_2(c + d) - \xi_2].$$

At $q = -1$,

$$N_{23} = (2(a + b) - 1)(1 + p),$$

so strictness gives $p = -1$, the base ray r_5 . The second factor is not identically zero: if $\xi_0 \neq 0$, its coefficient of q is nonzero; if $\xi_0 = 0$, then $\xi_2 \neq 0$ and its constant term $\xi_2(2(c + d) - 1)$ is nonzero.

For fixed q , N_{23} is linear in p , with coefficient $q + 2(a + b)$ and constant term $2(c + d)q + 2e$. Both could vanish only if

$$2(a + b)(c + d) = a + b + c + d - \frac{1}{2},$$

equivalently

$$(2(a + b) - 1)(2(c + d) - 1) = 0,$$

which strictness excludes. Thus there is at most one nonbase direct preimage.

Finally, $\omega(x) = 0$ is exactly the union of the four divisors just treated. Their zero-coordinate intersections were included before each scale choice: $x_2 = x_3 = 0$ gives r_1, r_2 , while $x_0 = x_1 = 0$ gives r_3, r_4 . All other intersections remain within the direct/cross alternatives above; none of those arguments divides by a target coordinate, and the cases $\xi_i = 0$ were retained in the resultant analysis. The four analyses therefore exhaust every exceptional intersection and prove projective injectivity away from the five-ray base locus.

H Bounded transportation and the rank-zero simulation

For three ternary labels, let $c_{ij} = c_{ji} \geq 0$, $c_{ii} = 0$, and assume

$$\sum_{j \neq i} c_{ij} = a_i + b_i, \quad \sum_i a_i = \sum_i b_i = \frac{1}{2}.$$

Seek q_{ij} with row sums b_i , column sums a_j , and $0 \leq q_{ij} \leq c_{ij}$. Put

$$q_{ij} = \frac{c_{ij}}{2} + f_{ij}, \quad f_{ij} = -f_{ji},$$

$$d_i = \frac{b_i - a_i}{2}, \quad d_1 + d_2 + d_3 = 0.$$

Set

$$f_{12} = t, \quad f_{13} = d_1 - t, \quad f_{23} = d_2 + t.$$

The row and column equations are automatic. Capacity is equivalent to

$$t \in I_{12} \cap I_{13} \cap I_{23},$$

where

$$I_{12} = [-c_{12}/2, c_{12}/2],$$

$$I_{13} = [d_1 - c_{13}/2, d_1 + c_{13}/2],$$

$$I_{23} = [-d_2 - c_{23}/2, -d_2 + c_{23}/2].$$

The inequalities

$$|d_i| \leq \frac{a_i + b_i}{2} = \frac{c_{ij} + c_{ik}}{2}$$

show that every pair of intervals intersects, including zero-capacity endpoints. Three closed intervals on the real line with pairwise intersections have a common point: the largest left endpoint does not exceed the smallest right endpoint. Any common t gives the required bounded transportation matrix.

The deterministic branch construction in proposition 9.2 then reproduces all four input blocks simultaneously. The supplied rational simulator independently checks the construction on an interior symmetric instance; generality follows from the interval proof, not from that one executable example.

I Verification boundary

The exact verifier checks:

1. dense and sparse Bell coefficients and both explicit strategies;
2. POVM normalization, rank-one factorizations, exact Bell values, and discrimination dual certificates;
3. the determinant, discriminant, robust-comparison, deficit, and complete-square identities in the projective bound;
4. the Lorentz metric identities, quadratic map, generic inverse, all exceptional resultants and factorizations;

5. an exact pure-state conformal-Lorentz instance and the weighted Hessian square completion;
6. independence witnesses for the five incidence constraints;
7. bounded-flow identities and an independent rank-zero simulation.

The scripts do not replace the manuscript arguments establishing compactness, extremal reduction, cone circuits, physical completeness, multiplier signs, dimension bounds, or exhaustion of cases. Some symbolic checks are exact regression instances rather than formal proofs of general matrix identities; the corresponding general identities are derived above.

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Table 2: Notation used in the separation and equality arguments.

Symbol	Meaning
Q_2^X	Raw behavior set from one fixed-qubit strategy of type $X \in \{\text{POVM}, \text{PVM}\}$
$\overline{Q}_2^X$	Shared-randomness convex hull $\text{conv } Q_2^X$
$\beta_X(c)$	Support value of Bell tensor c on the convexified set
$\mathcal{B}, S, T$	The 3×2 Bell functional, its CHSH term, and its auxiliary term
Ξ	2×2 coefficient matrix of a pure two-qubit state
Θ_Ξ, L_Ξ	Steering map $N \mapsto \Xi N^T \Xi^\dagger$ and its Lorentz-coordinate matrix
J	Minkowski matrix $\text{diag}(1, -1, -1, -1)$
E_A, E_B	Lorentz-coordinate frames formed by four local effects
g, h, Q	Alice and Bob Gram metrics and $Q = g^{-1}$
$r_1, \dots, r_5, u$	Coefficient rays and their common normalization vector
P	4×4 block of joint probabilities in the two local effect bases
F_j	Null-incidence constraint $(Pr_j)^T g^{-1} (Pr_j) = 0$
B_c	Matrix representing Bell tensor c in the incidence coordinates
λ, Λ	Determinant multipliers and $\Lambda = \sum_j \lambda_j r_j r_j^T$
$\mathcal{D}, K$	Metric differential and $K = \ker \mathcal{D}^T$
$\mathcal{H}_K, q$	Compatible W -space and weighted second form
$K_{j y}, \Gamma_y$	Local score operators and finite-POVM dual operators