No universal death–birth amplifier on a finite weighted population structure

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Abstract

We ask whether a finite loopless weighted population structure can strictly increase uniformly initialized single-mutant fixation probability, relative to the same-order complete graph, under death–birth updating for every mutant fitness $r > 1$. For complete directed support we derive the first $1/r$ correction exactly. Its excess loss is a sum of squares over incoming weight columns; equality holds precisely when every incoming column is constant, in which case the entire process is the complete-graph baseline. Combining this with the known transience theorem for strongly connected noncomplete supports and a source-component bound for non-strong supports proves that no finite directed weighting amplifies dB fixation for every $r > 1$. For undirected graphs we also give the exact incomplete-support strong limit. Finally, we classify positive weighted triangles and two maximally symmetric nontrivial families on four vertices at every beneficial fitness: every nonuniform member is a strict dB suppressor. These fixed-graph results do not settle the reversed asymptotic order in which the population threshold may depend on fitness.

Author summary

Evolution in a structured population depends on who can replace whom. A network is called an amplifier when it makes a beneficial mutation more likely to take over than it would be in a well-mixed population. We ask whether one fixed finite weighted network can amplify every possible beneficial mutation under death–birth replacement. It cannot. For complete directed networks, we calculate the first strong-selection correction and show that its deficit is an explicit sum of nonnegative squares. Equality occurs only when every competition is dynamically identical to the well-mixed process. Combining this with the remaining support cases rules out every finite directed network. We also obtain exact classifications for all weighted triangles and for two natural symmetric four-vertex families: every nonuniform example suppresses selection at every beneficial fitness. The proofs are accompanied by exact symbolic transition builders and independent replay checks. A different large-population question remains open because its population-size threshold may depend on fitness; we state this quantifier distinction and a necessary support-degree condition explicitly.

1 Relation to prior bounds

Tkadlec et al. proved that every strongly connected noncomplete population structure—allowing directed and weighted edges—is eventually dB-suppressing as fitness grows [1]. Their source-to-target weight convention, loopless model, and uniform initialization agree with ours. Their theorem leaves one strongly connected possibility: a nonuniform weighted version of the complete directed graph. Our incoming-column sum of squares closes precisely that case and gives the sharp equality class. A short source-component argument below also removes the strong-connectivity assumption from the final fixed-graph statement.

Transient dB amplification is known and can be diagnosed at weak selection using coalescing random walks [2]. More recently, weighted modular graphs have been proved to amplify both Bd and dB selection on the fixed fitness interval $(1, 1.2)$ [3]. Those positive results are consistent with the obstruction here: our conclusion concerns a fixed finite graph required to amplify for every $r > 1$, and failure occurs in its strong-selection tail.

2 Model and result

Let $G = (V, W)$ be loopless, $|V| = n \geq 2$, and let $w_{uv} \geq 0$ be the weight with which source u competes to replace dead target v. Assume each incoming degree

$$d_v^- := \sum_{u \neq v} w_{uv}$$

is positive. Mutants have fitness $r > 1$ and residents fitness one. Under death–birth updating a uniformly chosen target dies, after which a source is chosen with probability proportional to fitness times w_{uv}. Write $\rho_{\mathrm{dB}}(G, r)$ for fixation probability averaged uniformly over singleton initial mutants. Multiplying one incoming column by a positive constant is dynamically irrelevant for dB.

Theorem 1 (directed complete-support correction)**.** *Suppose $n \geq 3$ and $w_{uv} > 0$ for every $u \neq v$. Put*

$$\mathcal{E}_{\mathrm{dir}}(G) = \sum_{v} \sum_{\substack{u<z \\ u,z \neq v}} \frac{(w_{uv} - w_{zv})^2}{w_{uv} w_{zv}}.$$

Then

$$\rho_{\mathrm{dB}}(K_n, r) - \rho_{\mathrm{dB}}(G, r) = \frac{\mathcal{E}_{\mathrm{dir}}(G)}{n^2(n-2)r} + O(r^{-2}).$$

Here the expansion is for fixed G as $r \to \infty$. The defect vanishes if and only if for each target v there is $c_v > 0$ with $w_{uv} = c_v$ for all $u \neq v$. In that case the full dB chain equals the K_n chain for every $r > 0$. For $n = 2$, arbitrary positive w_{12}, w_{21} also give $\rho_{\mathrm{dB}}(G, r) = 1/2$.

Proposition 2 (non-strong supports)**.** *If the directed positive-weight support is not strongly connected, then G is strictly dB-suppressing for all sufficiently large finite r.*

Corollary 3 (fixed directed closure)**.** *No finite loopless directed weighting with positive incoming degrees is a strict dB amplifier for every $r > 1$. More precisely, it is eventually dB-suppressing unless it has complete support and constant incoming columns; the latter class ties the baseline exactly.*

For symmetric weights, Theorem 1 has a sharper support-level companion. Let

$$s_i = |\{j : w_{ij} > 0\}|.$$

Proposition 4 (undirected support refinement)**.** *For every connected undirected weighted graph,*

$$\lim_{r\to\infty} \rho_{\mathrm{dB}}(G,r) = \frac{1}{n}\sum_i \frac{s_i}{s_i+1}.$$

Consequently an incomplete support has limiting deficit

$$\sum_i \frac{n-1-s_i}{n^2(s_i+1)} > 0.$$

For complete symmetric support, $\mathcal{E}_{\mathrm{dir}}$ is the incident-star defect and vanishes only when all off-diagonal weights are globally equal.

Corollary 3 rules out, in particular,

$$\exists N_0\ \forall N \geq N_0\ \forall r > 1: \quad \rho_{\mathrm{dB}}(G_N, r) > \rho_{\mathrm{dB}}(K_{|V(G_N)|}, r),$$

but it does not decide the reversed order $\forall r > 1\, \exists N_0(r)\, \forall N \geq N_0(r)$.

The two branches of the certificate are shown schematically in Figure 1.

incomplete support
$\rho_{\mathrm{dB}}(K_n, r) - \rho_{\mathrm{dB}}(G, r)$
has a positive limiting gap

complete support
$\rho_{\mathrm{dB}}(K_n, r) - \rho_{\mathrm{dB}}(G, r)$
$= \mathcal{E}_{\mathrm{dir}}(G)/(n^2(n-2)r) + O(r^{-2})$

Figure 1: The undirected support refinement. The dashed segment on the left is a missing support edge. In the directed complete-support theorem, the right branch measures variation within incoming columns.

3 Complete-graph baselines

We derive both baselines directly. Let ϕ_k denote fixation probability on K_n from k mutants, and let p_k^+ and p_k^- be the probabilities of increasing and decreasing k. The first-step equations give

$$p_k^+(\phi_{k+1} - \phi_k) = p_k^-(\phi_k - \phi_{k-1}).$$

If $\gamma_k = p_k^-/p_k^+$, summing successive increments subject to $\phi_0 = 0$ and $\phi_n = 1$ gives

$$\phi_1 = \left(1 + \sum_{j=1}^{n-1}\prod_{k=1}^{j}\gamma_k\right)^{-1}.$$

For Bd updating, $\gamma_k = 1/r$, hence

$$\rho_{\mathrm{Bd}}(K_n, r) = \frac{1-r^{-1}}{1-r^{-n}}. \tag{1}$$

For dB updating, direct counting gives

$$p_k^+ = \frac{n-k}{n} \frac{rk}{rk+n-k-1}, \qquad p_k^- = \frac{k}{n} \frac{n-k}{r(k-1)+n-k}.$$

Writing $A_k = n-1+(r-1)k$, one has

$$\gamma_k = \frac{A_k}{rA_{k-1}}, \qquad \prod_{k=1}^{j} \gamma_k = \frac{A_j}{(n-1)r^j}.$$

The finite sum telescopes to the exact formula

$$\rho_{\mathrm{dB}}(K_n, r) = \frac{n-1}{n} \frac{1-r^{-1}}{1-r^{-(n-1)}}. \tag{2}$$

For $n \geq 3$ this has the strong-selection expansion

$$\rho_{\mathrm{dB}}(K_n, r) = \frac{n-1}{n} - \frac{n-1}{nr} + O(r^{-2}). \tag{3}$$

We will use the following elementary finite-chain perturbation fact to make the $r \to \infty$ passage precise.

Lemma 5 (finite-state perturbation)**.** *Let $P(\varepsilon)$ be a finite family of transition matrices continuous at $\varepsilon = 0$. Suppose that, from a specified initial state, all states reachable before absorption at $\varepsilon = 0$ lie in a set $\mathcal{R}$, the transient restriction $Q_{\mathcal{R}}(0)$ has spectral radius less than one, and the one-step probability of leaving $\mathcal{R}$ is $O(\varepsilon)$. Then the absorption probabilities converge to those of $P(0)$. If $\mathcal{R}$ is the full transient state space and the entries are analytic, then the absorption probabilities are analytic near zero.*

Proof. The inverse $(I - Q_{\mathcal{R}}(\varepsilon))^{-1}$ exists and remains bounded for all sufficiently small ε. Multiplying it by the vector of one-step leakage probabilities shows that the probability of leaving $\mathcal{R}$ before absorption is $O(\varepsilon)$. The absorption equations within $\mathcal{R}$ therefore converge to the nonsingular limiting system; behavior after leakage can change a hitting probability by at most the leakage probability. In the full-space analytic case, $(I - Q(\varepsilon))^{-1}$ is analytic by the adjugate formula. □

4 Noncomplete directed supports

We first prove Proposition 2. Consider the condensation DAG of the directed positive-weight support. If it has two source strongly connected components, then at least one starts entirely resident and cannot receive mutant offspring from outside, so singleton fixation is impossible. Suppose instead it has a unique source component $C \subsetneq V$. An initial mutant outside C cannot fixate. For $i \in C$, let $s_i^+ = |\{v : w_{iv} > 0\}|$. While i is the only mutant, write $p_{iv}(r) \leq 1$ for its conditional chance to replace target v when that target dies. After self-loops are removed, the chance of a first gain before the death of i is

$$\frac{\sum_{v:w_{iv}>0} p_{iv}(r)}{1 + \sum_{v:w_{iv}>0} p_{iv}(r)} \leq \frac{s_i^+}{s_i^+ + 1} \leq \frac{n-1}{n}.$$

Fixation requires this gain. Uniform initialization consequently gives

$$\rho_{\mathrm{dB}}(G, r) \leq \frac{1}{n} \sum_{i \in C} \frac{s_i^+}{s_i^+ + 1} \leq \frac{(n-1)^2}{n^2}. \tag{4}$$

The baseline tends to $(n-1)/n$, leaving the positive limiting gap $(n-1)/n^2$. This proves the proposition.

If the support is strongly connected but noncomplete, Theorem 1 of Tkadlec et al. gives a threshold $r^* \le 2n^2$ beyond which strict suppression holds [1]. If it is complete, Theorem 1 gives eventual strict suppression unless all incoming columns are constant, in which case columnwise normalization makes the chain exactly K_n. These three cases prove Corollary 3.

5 Undirected incomplete positive-weight support

Fix an initial mutant vertex i. At $r = \infty$, death of i causes extinction. Death of any one of its s_i support-neighbors is followed by mutant reproduction with probability one and creates an adjacent mutant pair. Deaths at nonneighbors are self-loops. Conditioning on the first state change therefore gives probability $s_i/(s_i+1)$ of reaching a pair before extinction.

Once an adjacent pair has formed, the mutant set remains support-connected. Every mutant has a mutant neighbor and hence cannot be replaced by a resident in the limiting chain. Whenever a boundary resident dies it becomes mutant. Connectedness of the support implies monotone growth to fixation almost surely. Thus

$$\lim_{r\to\infty} \rho_{\mathrm{dB}}(G,r) = \frac{1}{n}\sum_i \frac{s_i}{s_i+1}.$$

Subtracting this from $(n-1)/n$ gives the exact termwise certificate below.

$$\frac{n-1}{n} - \frac{1}{n}\sum_i \frac{s_i}{s_i+1} = \sum_i \frac{n-1-s_i}{n^2(s_i+1)}. \tag{5}$$

Every denominator is positive. If an edge is missing, at least one numerator is positive, proving Proposition 4.

For completeness, the limiting argument is stable: from a singleton, every limiting state reached before absorption is a connected mutant set. The finite limiting transient chain absorbs almost surely, so its fundamental matrix is nonsingular. For finite $r = 1/\varepsilon$, a transition that leaks from this set requires a resident to replace a mutant having a mutant neighbor, and has probability $O(\varepsilon)$ in each fixed state. Lemma 5 therefore proves convergence of the finite-r hitting probabilities, including the effect of all paths after such leakage.

6 Complete support and the sum-of-squares correction

Assume $n \ge 3$ and $w_{uv} > 0$ for all $u \ne v$. Put

$$\varepsilon = r^{-1}, \qquad a_{uv} = \frac{d_v^- - w_{uv}}{w_{uv}}.$$

Let $q_S(\varepsilon)$ be extinction probability from mutant set S. At $\varepsilon = 0$, every set with at least two mutants fixates, while $q_{\{i\}}(0) = 1/n$. The limiting transient chain absorbs almost surely, so Lemma 5 shows that the hitting probabilities are analytic near $\varepsilon = 0$.

For later use, we make the claimed first-order vanishing explicit. Write $P_{ST}(\varepsilon)$ for the transition probability from S to T. Every entry is analytic at zero: after multiplying mutant fitnesses by ε, each nontrivial denominator has a strictly positive constant term, while extinction from a singleton mutant death has probability exactly $1/n$. At $\varepsilon = 0$, a singleton either becomes extinct or creates a pair, and every state with at least two mutants then follows a pure-birth chain (apart from self-loops) to fixation. Hence the full limiting transient matrix has spectral radius less than one. The adjugate formula in Lemma 5 proves that every q_S is analytic near zero.

Differentiate the exact first-step system:

$$q'_S(0) = \sum_T P'_{ST}(0)q_T(0) + \sum_T P_{ST}(0)q'_T(0). \tag{6}$$

If S is proper and $3 \le m = |S| \le n-1$, a first-order loss removes only one mutant and therefore lands in a state with at least two mutants; all loss, self-loop, and gain destinations consequently have zeroth-order extinction probability zero. The first sum in (6) vanishes. In the limiting chain, $P_{SS}(0) = m/n$ and $P_{S,S\cup\{v\}}(0) = 1/n$ for each $v \notin S$, so

$$(n-m)q'_S(0) = \sum_{v\notin S} q'_{S\cup\{v\}}(0). \tag{7}$$

When $n \ge 4$, the right side is zero for $m = n-1$ because $q_V \equiv 0$; downward induction in mutant-set size then proves the assertion for all proper states with at least three mutants. For $n \le 3$ there are no such proper states, and $q'_V(0) = 0$ in every dimension. Thus $q'_S(0) = 0$ for every $|S| \ge 3$. Analyticity therefore gives

$$q_S(\varepsilon) = O(\varepsilon^2), \qquad |S| \ge 3. \tag{8}$$

For $S = \{i,j\}$, loss of mutant target i occurs with probability

$$\frac{1}{n}\frac{\varepsilon(d_i^- - w_{ji})}{w_{ji} + \varepsilon(d_i^- - w_{ji})} = \frac{\varepsilon a_{ji}}{n} + O(\varepsilon^2),$$

while loss of target j contributes $\varepsilon a_{ij}/n + O(\varepsilon^2)$. The leading self-loop probability is $2/n$ and the leading total gain probability is $(n-2)/n$. Since a resulting singleton has extinction probability $1/n + O(\varepsilon)$, the exact first-step equation gives, on putting

$$b_{ij} = \frac{a_{ij} + a_{ji}}{n(n-2)},$$

$$q_{\{i,j\}}(\varepsilon) = b_{ij}\varepsilon + O(\varepsilon^2). \tag{9}$$

From singleton $\{i\}$, death of resident target v creates $\{i,v\}$ with probability

$$t_{iv} = \frac{1}{n}\frac{w_{iv}}{w_{iv} + \varepsilon(d_v^- - w_{iv})} = \frac{1}{n}(1 - a_{iv}\varepsilon) + O(\varepsilon^2).$$

Death of i causes extinction with probability exactly $1/n$; every other outcome is a self-loop. Hence

$$\left(\frac{1}{n} + \sum_{v\neq i} t_{iv}\right) q_{\{i\}} = \frac{1}{n} + \sum_{v\neq i} t_{iv} q_{\{i,v\}}.$$

Substitution of (9) yields the useful vertexwise intermediate formula. Define

$$O_i = \sum_{v\neq i} a_{iv}, \qquad I_i = \sum_{u\neq i} a_{ui}.$$

Since $\sum_{j\neq i} b_{ij} = (O_i + I_i)/(n(n-2))$, expansion of the preceding singleton equation gives

$$q_{\{i\}}(\varepsilon) = \frac{1}{n} + \frac{\varepsilon}{n^2}\left[O_i + \frac{O_i + I_i}{n-2}\right] + O(\varepsilon^2). \tag{10}$$

Now $\sum_i O_i = \sum_i I_i = T_{\text{dir}}(G)$, and averaging (10) yields

$$T_{\text{dir}}(G) = \sum_v \sum_{u\neq v} a_{uv}, \qquad A_{\text{dir}}(G) = \frac{T_{\text{dir}}(G)}{n^2(n-2)},$$

$$\rho_{\text{dB}}(G,r) = \frac{n-1}{n} - \frac{A_{\text{dir}}(G)}{r} + O(r^{-2}). \tag{11}$$

For the unit complete graph, $a_{ij} = n-2$ and $T_{\text{dir}}(K_n) = n(n-1)(n-2)$. The difference admits the exact identity

$$\begin{aligned} T_{\text{dir}}(G) - n(n-1)(n-2) &= \sum_v \left[d_v^- \sum_{u\neq v} \frac{1}{w_{uv}} - (n-1)^2\right] \\ &= \sum_v \sum_{\substack{u<z \\ u,z\neq v}} \frac{(w_{uv} - w_{zv})^2}{w_{uv}w_{zv}} = \mathcal{E}_{\text{dir}}(G). \end{aligned} \tag{12}$$

All denominators in (12) are positive. Equality requires every incoming column to be constant. Independent positive rescaling of incoming columns cancels from every dB competition, so the equality class has exactly the complete-graph chain. Under symmetry, the column constants are coupled and force one global edge weight. Equations (3), (11), and (12) prove Theorem 1.

7 Beneficial-fitness classification on three vertices

We now specialize to an undirected triangle with positive edge weights (a, b, c). Let F_i be fixation probability when i is the unique mutant and G_i when i is the unique resident. For distinct i, j, k, put

$$u_{ij} = \frac{rw_{ij}}{rw_{ij} + w_{jk}}, \qquad v_{ij} = \frac{w_{ij}}{rw_{jk} + w_{ij}}.$$

Deleting holding probabilities from the literal dB first-step equations gives

$$(1 + u_{ij} + u_{ik})F_i = u_{ij}G_k + u_{ik}G_j, \tag{13}$$

$$(1 + v_{ij} + v_{ik})G_i = 1 + v_{ij}F_k + v_{ik}F_j. \tag{14}$$

The coefficient matrix M has nonpositive off-diagonal entries and each diagonal entry equals one plus the sum of the absolute off-diagonal entries. It is therefore strictly diagonally dominant, and the homotopy $I + t(M - I)$ shows $\det M > 0$.

Set $s_1 = a + b + c$, $s_2 = ab + ac + bc$, and $s_3 = abc$, and define

$$\begin{aligned} B_5 &= 12s_1s_2s_3 - 36s_3^2, \\ B_4 &= 12s_1^3s_3 - 56s_1s_2s_3 + 12s_2^3 + 72s_3^2, \\ B_3 &= -24s_1^3s_3 + 12s_1^2s_2^2 + 80s_1s_2s_3 - 24s_2^3 - 90s_3^2, \\ P &= 9s_3^2(r^6 + 1) + B_5(r^5 + r) + B_4(r^4 + r^2) + B_3r^3. \end{aligned}$$

If $L = \prod_{p \neq q}(rp+q)$ over $p, q \in \{a, b, c\}$, exact expansion of (13)–(14) gives $\det M = 3P/L$. Thus $P > 0$ for positive weights and fitness.

Define

$$\begin{aligned}
A &= 3s_3(s_1 s_2 - 9s_3),\\
D &= 12s_1^3 s_3 - 45s_1 s_2 s_3 + 4s_2^3 - 27s_3^2,\\
E &= 4s_2(3s_1^2 s_2 - 3s_1 s_3 - 8s_2^2),\\
H &= A(r-1)^4 + Dr(r-1)^2 + Er^2.
\end{aligned}$$

Solving the six equations and averaging the three F_i gives the exact rational comparison

$$\rho_{\mathrm{dB}}(G, r) - \rho_{\mathrm{dB}}(K_3, r) = -\frac{r(r-1)H}{3(r+1)P}. \tag{15}$$

For a sign certificate, put

$$U = s_1^2 - 3s_2, \qquad V = s_2^2 - 3s_1 s_3, \qquad W = s_1 s_2 - 9s_3, \qquad Z = s_2^3 - 27s_3^2.$$

Writing $X = ab, Y = ac, Z_0 = bc$, direct expansion yields

$$\begin{aligned}
2U &= (a-b)^2 + (a-c)^2 + (b-c)^2,\\
2V &= (X-Y)^2 + (X-Z_0)^2 + (Y-Z_0)^2,\\
W &= c(a-b)^2 + b(a-c)^2 + a(b-c)^2,\\
Z &= s_2 V + 3\{Z_0(X-Y)^2 + Y(X-Z_0)^2 + X(Y-Z_0)^2\}.
\end{aligned}$$

Finally, the coefficient identities

$$A = 3s_3 W, \qquad D = 12s_1 s_3 U + 3s_2 V + Z, \qquad E = 4s_2(3s_2 U + V)$$

show $A, D, E \geq 0$, with $E > 0$ unless $a = b = c$.

Theorem 6 (weighted-triangle classification)**.** *For every positive undirected triangle and every $r > 1$,*

$$\rho_{\mathrm{dB}}(G, r) \leq \rho_{\mathrm{dB}}(K_3, r) = \frac{2r}{3(r+1)}.$$

Equality holds if and only if $a = b = c$; every nonuniform weighting is a strict dB suppressor throughout the beneficial-fitness interval.

Proof. All factors in the denominator of (15) are positive. For nonuniform weights, $U > 0$, hence $E > 0$ and $H \geq Er^2 > 0$; the difference is therefore negative for $r > 1$. Uniform weights give $H = 0$ and the baseline chain exactly. □

8 Two symmetric complete-support families on four vertices

The triangle result extends to the two most symmetric nontrivial orbit families on four vertices. First let $G_{13}(x)$ have one core, three satellites, unit core–satellite weights, and satellite–satellite weight $x > 0$. Second let $G_{22}(x, y)$ have two vertex pairs, internal-pair weights $x, y > 0$, and four unit cross-pair weights. These are respectively the full $S_1 \times S_3$ and $S_2 \times S_2$ families after common weight scaling.

Figure 2: The two symmetric weighted K_4 families. Every unlabeled core–satellite or cross-pair edge has unit weight; every edge in a displayed orbit has the indicated common weight.

For reference, a two-class graph with class sizes p, q, internal weights α, β, and cross weight γ is strongly lumpable by the mutant counts (i, j). Directly from the dB rule, its four state-changing probabilities are

$$\begin{aligned}
P_{i,j}^{i+1,j} &= \frac{p-i}{p+q} \frac{r(i\alpha + j\gamma)}{r(i\alpha + j\gamma) + (p-i-1)\alpha + (q-j)\gamma}, \\
P_{i,j}^{i-1,j} &= \frac{i}{p+q} \frac{(p-i)\alpha + (q-j)\gamma}{r((i-1)\alpha + j\gamma) + (p-i)\alpha + (q-j)\gamma}, \\
P_{i,j}^{i,j+1} &= \frac{q-j}{p+q} \frac{r(j\beta + i\gamma)}{r(j\beta + i\gamma) + (q-j-1)\beta + (p-i)\gamma}, \\
P_{i,j}^{i,j-1} &= \frac{j}{p+q} \frac{(q-j)\beta + (p-i)\gamma}{r((j-1)\beta + i\gamma) + (q-j)\beta + (p-i)\gamma}.
\end{aligned} \tag{16}$$

Terms with zero target multiplicity are omitted. These formulas prove lumpability rather than assuming it.

For the $1+3$ family, put

$$\begin{aligned}
F_{13} &= 2r^4x^2 + 2r^4x + 11r^3x^2 + 14r^3x + 3r^3 + 21r^2x^2 + 29r^2x + 12r^2 \\
&\quad + 16rx^2 + 22rx + 8r + 4x^2 + 5x + 1, \\
P_{13} &= 8x^2(x+1)(r^6+1) + (6x^4 + 36x^3 + 46x^2 + 16x)(r^5 + r) \\
&\quad + (27x^4 + 73x^3 + 85x^2 + 59x + 12)(r^4 + r^2) \\
&\quad + (42x^4 + 90x^3 + 106x^2 + 66x + 24)r^3.
\end{aligned}$$

Solving the six transient equations from (16) gives

$$\rho_{\mathrm{dB}}(G_{13}(x), r) - \rho_{\mathrm{dB}}(K_4, r) = -\frac{3r^2(r-1)(x-1)^2 F_{13}}{4(r^2+r+1)P_{13}}. \tag{17}$$

Every coefficient of F_{13} and P_{13} is positive.

The $2+2$ numerator is larger, but it has a short domain-adapted certificate. Exact elimination of the seven transient equations gives

$$\rho_{\mathrm{dB}}(G_{22}(x, y), r) - \rho_{\mathrm{dB}}(K_4, r) = -\frac{r^2(r-1)H_{22}(x, y, r)}{4(r^2+r+1)P_{22}(x, y, r)}. \tag{18}$$

The reduced polynomial P_{22} has 123 positive integer monomials. There is also a structural denominator certificate. If M_{22} is the seven-state holding-step-free transient coefficient matrix defined by

(16), exact expansion gives

$$\begin{aligned}
\det M_{22} &= \frac{P_{22}}{128 L_{22}},\\
L_{22} &= (2r+x)(2r+y)(rx+2)(ry+2)(r+x+1)(r+y+1)\\
&\qquad \times (rx+r+1)(ry+r+1).
\end{aligned} \tag{19}$$

The finite quotient chain absorbs almost surely, so M_{22} is a nonsingular M-matrix and its determinant is positive. Hence $P_{22} > 0$ independently of its coefficient listing.

For the numerator, set

$$g = \sqrt{xy} > 0, \qquad d = (\sqrt{x} - \sqrt{y})^2 \geq 0, \qquad t = r - 1 > 0.$$

Symmetry in x, y makes the substitutions $x + y = 2g + d$ and $xy = g^2$ exact. The polynomial identity is

$$H_{22} = C_0(g,t) + C_1(g,t)d + C_2(g,t)d^2 + C_3(g,t)d^3 + C_4(g,t)d^4, \tag{20}$$

where

$$\begin{aligned}
C_0 &= 2t(g-1)^2(g+1)(t+1)R_0,\\
R_0 &= (2g^2+2g)t^4 + (g^3+10g^2+21g+6)t^3\\
&\quad + (3g^3+26g^2+61g+38)t^2 + (4g^3+32g^2+80g+64)t\\
&\quad + 2g^3+16g^2+40g+32,\\
C_1 &= 2(g^4+4g^3-2g^2+4g+1)t^6 + (11g^4+108g^3+76g^2+60g+17)t^5\\
&\quad + 2(40g^4+288g^3+393g^2+176g+19)t^4\\
&\quad + (16g^5+331g^4+1704g^3+2766g^2+1360g+39)t^3\\
&\quad + 2(28g^5+337g^4+1426g^3+2399g^2+1378g+12)t^2\\
&\quad + 2(g+2)(32g^4+263g^3+722g^2+661g+2)t\\
&\quad + 24g(g+2)(g+4)(g^2+4g+5),\\
C_2 &= 2(g^2+1)t^6 + 3(9g^2+26g+5)t^5 + 2(18g^3+120g^2+321g+44)t^4\\
&\quad + 2(2g^4+105g^3+525g^2+1071g+170)t^3\\
&\quad + (14g^4+474g^3+2201g^2+3648g+689)t^2\\
&\quad + 2(8g^4+240g^3+1080g^2+1587g+331)t\\
&\quad + 6(g^4+30g^3+133g^2+186g+40),\\
C_3 &= 13t^5 + (6g^2+48g+107)t^4 + (35g^2+312g+357)t^3\\
&\quad + (79g^2+744g+608)t^2 + (80g^2+768g+529)t + 6(5g^2+48g+31),\\
C_4 &= 6t^4 + 39t^3 + 93t^2 + 96t + 36.
\end{aligned}$$

Every displayed coefficient is positive except for the superficially negative monomial in the leading coefficient of C_1, and

$$g^4 + 4g^3 - 2g^2 + 4g + 1 = (g^2-1)^2 + 4g(g^2+1) > 0.$$

Thus $C_1, C_2, C_3, C_4 > 0$ and $C_0 \geq 0$. If $d > 0$, then $C_1 d > 0$; if $d = 0$, then $H_{22} = C_0$, which vanishes only at $g = 1$. This proves the following exact family classification.

Theorem 7 (symmetric weighted K_4 families)**.** *For every $r > 1$,*

$$\rho_{\mathrm{dB}}(G_{13}(x), r) \leq \rho_{\mathrm{dB}}(K_4, r), \qquad \rho_{\mathrm{dB}}(G_{22}(x, y), r) \leq \rho_{\mathrm{dB}}(K_4, r).$$

Equality holds only at $x = 1$ in the first family and $x = y = 1$ in the second. Consequently every nonuniform member of either family is a strict dB suppressor throughout the beneficial-fitness interval.

Theorem 7 is not a classification of the unrestricted six-edge weighted K_4 family. That problem remains open.

9 Exact verification

Four exact verification paths accompany the proof. The original undirected verifier constructs every subset-state transition over $\mathbb{Q}(r)$, checks row normalization and representative positivity certificates, verifies complete-graph lumpability, solves the absorption equations, and extracts strong-selection coefficients. A separate directed implementation uses the literal source-to-target rule on genuinely asymmetric $n = 3, 4$ matrices. It also checks independent incoming-column rescaling and a column-uniform, row-nonuniform negative control that falsifies the wrong row-oriented defect. For Theorem 6, one script derives the six equations and all polynomial identities, while a full eight-state transition implementation and a hostile no-import replay check the resulting comparison. For Theorem 7, a separate orbit derivation checks (17)–(20); a full subset-state implementation checks the lumping and rational formulas, and a hostile audit independently repeated the symbolic 14-state solves.

These computations validate implementations and replay exact consequences; finite testing does not discharge a universal quantifier. The complete- support quantifier is proved by (6)– (10) and (12); the undirected refinement uses (5); and the triangle theorem uses the displayed homogeneous identities; the symmetric K_4 families use the coefficient certificate (20). The remaining directed-support branches follow from (4) and the cited noncomplete-support theorem.

As one nontrivial replay, take a weighted triangle with edge weights $(w_{12}, w_{13}, w_{23}) = (2, 3, 5)$. Exact elimination gives

$$\rho_{\mathrm{dB}}(G, r) - \rho_{\mathrm{dB}}(K_3, r) = -\frac{r(r-1)P_{235}(r)}{9(r+1)Q_{235}(r)},$$

where

$$\begin{aligned} P_{235}(r) &= 900r^4 + 5491r^3 + 9290r^2 + 5491r + 900, \\ Q_{235}(r) &= 675r^6 + 6600r^5 + 21791r^4 + 31768r^3 \\ &\qquad + 21791r^2 + 6600r + 675. \end{aligned}$$

Every displayed denominator factor is positive and the numerator is strictly negative for $r > 1$. The verifier also recovers the strong coefficient $22/27 > 2/3$, and recovers $343/320 > 3/4$ for the complete four-vertex weighting with lexicographic edge weights $(1, 2, 3, 4, 5, 6)$. For incomplete support it recovers limits $7/12$ for the four-vertex path and $9/16$ for the four-vertex star, against the baseline $3/4$.

These calculations are checks, not evidence used to extend a finite search.

10 Finite versus asymptotic statements

The complete-support expansion in Theorem 1, the source-component bound, and the cited non-complete theorem are all fixed-graph statements. They imply that for every G not dB-equivalent to the complete graph there is a finite threshold R_G such that

$$\rho_{\mathrm{dB}}(G,r) < \rho_{\mathrm{dB}}(K_n,r) \qquad (r > R_G).$$

No claim of a graph-independent threshold is needed. The all-r amplifier property quantifies over every finite $r > 1$ for each graph, so the existence of one such suppressing tail already rules it out. The equality case is exact, not asymptotic: independent incoming-column scaling cancels from every dB parent competition, so a column-uniform complete directed weighting has precisely the baseline chain for all finite r.

This argument does not interchange the limits in n and r. In particular, it leaves open the distinct possibility that one fitness-independent family might amplify at every fixed $r > 1$ once $N \geq N_0(r)$, with its individual strong-selection thresholds tending to infinity.

There is nevertheless a useful necessary condition. For a mutant set S and dead target v, the mutant replacement probability

$$\frac{r\sum_{u\in S} w_{uv}}{r\sum_{u\in S} w_{uv} + \sum_{u\notin S} w_{uv}}$$

is nondecreasing both in r and under inclusion of S. A common dead target and uniform threshold therefore couple the dB chains monotonically. Combining fitness monotonicity with Proposition 4 gives, for every finite connected undirected graph,

$$\rho_{\mathrm{dB}}(G,r) \leq 1 - \frac{1}{n}\sum_i \frac{1}{s_i+1}. \tag{21}$$

Proposition 8 (necessary asymptotic support condition)**.** *Let G_n be finite connected undirected weighted graphs with $|V(G_n)| = n \to \infty$. If G_n is eventually a dB amplifier at every fixed $r > 1$, then*

$$\frac{1}{n}\sum_i \frac{1}{s_i+1} \longrightarrow 0.$$

Consequently the support degree of a uniformly chosen vertex tends to infinity in probability.

Proof. At any fixed $R > 1$, eventual amplification and (21) imply

$$\limsup_{n\to\infty} \frac{1}{n}\sum_i \frac{1}{s_i+1} \leq \lim_{n\to\infty}\bigl(1-\rho_{\mathrm{dB}}(K_n,R)\bigr) = \frac{1}{R}.$$

Letting the fixed number R grow proves the first claim. For every fixed K, the fraction of vertices with $s_i \leq K$ is at most $(K+1)$ times the displayed average, proving the second. □

This excludes uniformly bounded-support families. It excludes repeated bounded-degree gadgets when a positive fraction of vertices retains bounded total support degree after the gadgets are connected; arbitrarily weak support-completion edges can fall outside this criterion. A companion exact report also proves suppression for fixed-class, positive-proportion dense equitable blow-ups with a fixed irreducible class kernel and unequal limiting weighted degrees. Diffuse asymptotically regular perturbations, mesoscopic modules, vanishing class proportions, and reducible limiting kernels remain open; no universal fixed-r obstruction is claimed.

Research and publication statements

Data and code availability.

No empirical data are used. The complete LaTeX source, exact transition builders, symbolic certificates, hostile-audit reports, and one-command reproduction target are archived with release `universal-db-obstruction-v1.0.0` at `github.com/AlecKriebel/Math`. The finite computations audit the encoded algebra; they are not substitutes for the universal proofs.

AI-assisted research disclosure.

Generative-AI systems were used substantively in exploration, derivation, adversarial proof analysis, software development, literature support, and manuscript preparation. Alec Kriebel determined the released scope and claims and accepts responsibility for the manuscript and verification materials. This preprint has not undergone external specialist peer review.

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