A transversal-capacity obstruction for a regular $R(5,5)$ endpoint

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Abstract

We give an elementary capacity inequality for hypergraph transversals indexed by the vertices of a graph. Applied to the cross-neighborhoods at a vertex of a hypothetical 18-regular $(5,5;43)$-graph, it excludes the endpoint

$$e(G[N(v)]) = 85.$$

On byte-pinned published $R(4,5;18)$ and $R(4,5;24)$ catalogs, the inequality directly excludes 61,939 of 62,382 fixed-side pairs. The remaining 443 attain equality. Uniqueness of a minimizing transversal then forces two adjacent cross-neighborhoods to coincide, contradicting the required two-column covering condition. Conditional on the published catalog-completeness and extremal-edge statements, every vertex of an 18-regular $(5,5;43)$-graph therefore lies in at most 84 triangles. A compact standard-library verifier reconstructs all finite calculations. This excludes one endpoint layer; it does not improve the known bounds on $R(5,5)$.

Verification disclaimer. This is an unreviewed AI-assisted research note released for expert checking. The human author is a complete amateur and cannot independently validate the mathematics. Exact computation verifies the encoded finite statements; it does not prove the external catalog-completeness assertions or establish literature priority.

1 Introduction

The diagonal Ramsey number $R(5,5)$ is the least n for which every graph on n vertices contains either a clique of order five or an independent set of order five. Exoo's 42-vertex construction gives the lower bound $R(5,5) \geq 43$ [1]. Angeltveit and McKay recently proved $R(5,5) \leq 46$ [7]. The exact value remains unknown.

Call a graph with neither a K_5 nor an independent five-set a $(5,5)$-graph. One possible route to a better upper bound is to assume a $(5,5)$-graph of order 43, root it at a vertex, and glue its neighborhood to its anti-neighborhood. The resulting two sides are smaller $R(4,5)$-graphs. This strategy has a long history: feasible-cone gluing was central to the proof that $R(4,5) = 25$ [3], and modern $R(5,5)$ computations combine catalog generation, local gluing, linear programming, and subgraph-count identities [2, 4, 5, 7].

The contribution here is narrow. For each allowed cross-neighborhood size, we compute the least possible number of independent triples missed by a cross-neighborhood that meets every independent four-set. Summing these minimum-miss penalties across all columns yields a necessary capacity inequality. At one regular degree-18 endpoint, the inequality eliminates almost every fixed-side pair; equality is rigid enough to eliminate the rest.

Theorem 1 (Endpoint theorem). *Assume the completeness and nonisomorphism statements for the published edge-85 $R(4,5;18)$ catalog and the complete $R(4,5;24)$ catalog. No 18-regular $(5,5)$-graph on 43 vertices has a vertex v with*

$$e(G[N(v)]) = 85.$$

Together with the published extremal bound $e(G[N(v)]) \le 85$, every vertex of such a graph lies in at most 84 triangles.

The theorem is catalog-conditional and endpoint-local. It does not exclude the other regular degree-18 layers, close another global branch, or alter the known range for $R(5,5)$.

2 A transversal-capacity inequality

Let U be a finite set, let $\mathcal{F}, \mathcal{Q} \subseteq 2^U$, and let J be a graph with $\alpha(J) \le a$. Assign a set $X_b \subseteq U$ to each $b \in V(J)$ so that

1. X_b meets every member of $\mathcal{F}$; and
2. $X_b \cup X_c$ meets every member of $\mathcal{Q}$ whenever $bc \in E(J)$.

For an integer s, define

$$q_s(U;\mathcal{F},\mathcal{Q}) = \min_{\substack{X \subseteq U,\ |X|=s \\ X \cap F \neq \varnothing\ \forall F \in \mathcal{F}}} \big|\{Q \in \mathcal{Q} : X \cap Q = \varnothing\}\big|, \tag{1}$$

with $q_s = +\infty$ if no size-s transversal of $\mathcal{F}$ exists.

Lemma 2 (Transversal capacity). *Every such assignment satisfies*

$$\sum_{b \in V(J)} q_{|X_b|}(U;\mathcal{F},\mathcal{Q}) \le a|\mathcal{Q}|. \tag{2}$$

Proof. For $Q \in \mathcal{Q}$, put

$$Z_Q = \{b \in V(J) : X_b \cap Q = \varnothing\}.$$

If two vertices of Z_Q were adjacent in J, their assigned sets would have a union disjoint from Q, contrary to the second hypothesis. Thus Z_Q is independent and $|Z_Q| \le a$. Double counting the incidences (b, Q) for which $X_b \cap Q = \varnothing$ gives

$$\sum_b q_{|X_b|} \le \sum_b |\{Q : X_b \cap Q = \varnothing\}| = \sum_Q |Z_Q| \le a|\mathcal{Q}|. \qquad \square$$

Corollary 3 (Equality rigidity). *If equality holds in* (2)*, then every X_b attains the minimum missed-$\mathcal{Q}$ count at its size, and every Z_Q has size a. In particular, suppose that the size-s minimizer is unique, $q_s > 0$, and the vertices b with $|X_b| = s$ induce an edge in J. Then no assignment satisfying the two hypotheses of Lemma 2 exists.*

Proof. Equality at the endpoints of

$$\sum_b q_{|X_b|} \le \sum_Q |Z_Q| \le a|\mathcal{Q}|$$

forces equality term by term in both inequalities. Thus each X_b minimizes at its size and every $|Z_Q| = a$. Under the additional hypotheses, the columns at the endpoints of an induced edge both equal the unique minimizer X^*. Since $q_s > 0$, some $Q \in \mathcal{Q}$ misses X^* and hence also misses the union of those two columns, contradicting the edge condition. $\square$

The same proof admits fixed nonnegative weights. If

$$q_s^w = \min_{\substack{X \subseteq U,\ |X|=s \\ X \cap F \neq \varnothing\ \forall F \in \mathcal{F}}} \sum_{\substack{Q \in \mathcal{Q} \\ X \cap Q = \varnothing}} w_Q,$$

then

$$\sum_b q^w_{|X_b|} \leq a \sum_{Q \in \mathcal{Q}} w_Q.$$

This weighted statement is an immediate template, not a tested claim about the remaining $R(5,5)$ layers.

3 Ramsey specialization

Let G be a hypothetical 18-regular $(5,5)$-graph on 43 vertices. Fix $v \in V(G)$ and put

$$A = G[N_G(v)], \qquad B = V(G) \setminus (N_G(v) \cup \{v\}), \qquad H = \overline{G[B]}.$$

Then $|A| = 18$, $|H| = 24$, and both A and H contain neither a K_4 nor an independent five-set. In particular, $\alpha(H) \leq 4$.

For $b \in B = V(H)$ define its cross-neighborhood

$$X_b = N_G(b) \cap V(A).$$

The absence of independent five-sets in G implies two local covering conditions:

1. X_b meets every independent four-set of A;
2. if $bc \in E(H)$, then $X_b \cup X_c$ meets every independent three-set of A.

Indeed, a missed independent four-set together with b, or a missed independent triple together with the G-nonedge bc, would be an independent five-set in G. The second rule is the classical two-column feasible-cone condition E_2 of McKay and Radziszowski [3]; the first is its immediate one-column forbidden-set analogue.

Regularity determines every column size:

$$18 = |X_b| + \deg_{G[B]}(b) = |X_b| + 23 - d_H(b),$$

and hence

$$|X_b| = d_H(b) - 5. \tag{3}$$

Let $\mathcal{I}_k(A)$ denote the independent k-sets of A, put $i_3(A) = |\mathcal{I}_3(A)|$, and specialize (1) to

$$q_s(A) = \min_{\substack{X \subseteq V(A),\ |X|=s \\ X \cap I \neq \varnothing\ \forall I \in \mathcal{I}_4(A)}} |\{Q \in \mathcal{I}_3(A) : X \cap Q = \varnothing\}|. \tag{4}$$

Lemma 2, with $J = H$ and $a = 4$, gives the necessary condition

$$\boxed{\sum_{b \in V(H)} q_{d_H(b)-5}(A) \leq 4 i_3(A).} \tag{5}$$

The two fixed-side edge counts are linked exactly. Summing degrees over A shows that the number of A–B edges is

$$18 \cdot 18 - 18 - 2e(A) = 306 - 2e(A).$$

On the B side, (3) gives

$$\sum_{b \in B} |X_b| = \sum_{b \in V(H)} (d_H(b) - 5) = 2e(H) - 120.$$

Equating these quantities yields

$$e(H) = 213 - e(A). \tag{6}$$

Thus $e(A) = 85$ is exactly the endpoint $e(H) = 128$.

4 The finite endpoint calculation

The external catalog inputs are the publisher-supplied graph6 files on McKay's Ramsey data page [8]. Angeltveit and McKay's 2026 paper records $E(4,5;18) = 85$, the 74 edge-85 order-18 graphs, and the 843 edge-128 order-24 graphs [7]. The relevant data files contain those 74 endpoint records and the complete 352,366-record $R(4,5;24)$ catalog. Completeness of the order-24 catalog is Theorem 2.1 of Angeltveit–McKay [5].

The exact input bytes used here are

$$\begin{aligned} \text{SHA256}(\texttt{r4518.85.g6}) &= \texttt{46abaee2572d06bba1e594554809d784be} \\ &\quad\ \texttt{60f8f60b9b0d3345b8bf3dd800810a}, \\ \text{SHA256}(\texttt{r45\_24.g6}) &= \texttt{83ca4028f206b2fa4315ef219b8c2c57} \\ &\quad\ \texttt{c7835209673dd8183d8fb4353bd4fdd0}. \end{aligned}$$

For every order-18 record A, the verifier enumerates all independent triples and four-sets, then every exact-size subset needed by the degree sequences of the selected H records. It retains precisely those subsets that meet every independent four-set and computes their missed-triple counts. This determines the exact profile values (4). Each of the $74 \cdot 843 = 62,382$ fixed-side pairs is then classified by (5). The profiles, degree data, and terminal equality conditions are invariant under relabeling. Thus every choice of side isomorphisms is covered; no cross-side vertex matching is an omitted parameter.

Classification	Pair count
Strict violation of the capacity inequality	61,939
Equality cases requiring terminal analysis	443
Pairs admitting a completion after terminal analysis	0
Total fixed-side pairs	62,382

The 61,939 strict violations cannot admit any cross-edge completion. The next section eliminates every equality pair without enumerating cross-edge matrices.

Theorem 4 (Finite pinned-catalog theorem)**.** *Let $\mathcal{A}_{85}$ be the 74 graphs encoded in the pinned edge-85 $R(4,5;18)$ endpoint file, and let $\mathcal{H}_{128}$ be the 843 edge-128 graphs selected from the pinned $R(4,5;24)$ file. For no pair*

$$(A, H) \in \mathcal{A}_{85} \times \mathcal{H}_{128}$$

does there exist a family $(X_b)_{b \in V(H)}$ satisfying the prescribed sizes (3), the one-column independent-four-set condition, and the two-column independent-triple condition.

5 Equality rigidity

All 443 equality pairs use the same order-18 graph A, at zero-based catalog index 50, and every associated H has degree sequence

$$10^8\,11^{16}.$$

For this exceptional A, exact enumeration gives

$$i_3(A) = 74, \qquad q_5(A) = 17, \qquad q_6(A) = 10.$$

Moreover, the size-six minimizer in (4) is unique; call it X^*. The capacity inequality is tight:

$$8q_5(A) + 16q_6(A) = 8 \cdot 17 + 16 \cdot 10 = 296 = 4i_3(A). \tag{7}$$

Suppose a cross-neighborhood assignment existed. The proof of Lemma 2 gives a chain

$$\sum_b q_{|X_b|}(A) \le \sum_{Q \in \mathcal{I}_3(A)} |Z_Q| \le 4i_3(A).$$

Equation (7) forces equality throughout. By Corollary 3, every column attains its individual profile minimum. In particular, every one of the sixteen degree-11 vertices of H has a size-six cross-neighborhood equal to the unique minimizer X^*.

Let D be those sixteen vertices. Each $b \in D$ has at most eight neighbors outside D, so it has at least three neighbors in D. Thus $H[D]$ contains an edge bc. But

$$X_b \cup X_c = X^*,$$

and X^* misses ten independent triples of A. This contradicts the two-column condition on the edge bc. All 443 equality pairs are therefore impossible, proving Theorem 4. The published catalog coverage and extremal-edge statements then give Theorem 1.

6 Verification and reproducibility

The computation is deterministic, solver-free, and uses only Python's standard library. The public bundle contains:

- the two byte-pinned source catalogs and their license attribution;
- the certificate producer and a separately implemented checker that imports no producer code;
- the complete 62,382-line classification stream;
- 17 focused semantic tests; and
- a one-command launcher that verifies file hashes, copies the release to a temporary directory, regenerates the frozen artifacts byte-for-byte, runs the checker, and runs the tests.

The corrected v1.0.1 normalized release archive has SHA-256

`de541d6c7ed8be496784397ea0ee3f1`
`b12c2b93cdbc42ba908160095c1d79cc4`.

A fresh copy downloaded from the public release returned `"valid": true`, reproduced every frozen artifact, and passed all 17 tests in under 70 seconds with peak resident memory below 140 MB. The release, source branch, and verification instructions are archived at

`https://github.com/AlecKriebel/Math/releases/tag/ramsey55-endpoint-capacity-v1.0.1`.

The verifier checks the exact supplied bytes, graph validity, Ramsey properties, edge filters, profile values, pair classifications, and equality witnesses. It does not independently regenerate the complete $R(4,5)$ catalogs. Catalog completeness and pairwise nonisomorphism remain external hypotheses.

7 Relation to prior work and scope

The two-column covering rule is the E_2 feasible-cone condition used by McKay and Radziszowski in their computation of $R(4,5)$ [3]; the one-column rule is its immediate forbidden-set analogue. The summation in Lemma 2 is an elementary incidence double count. Global neighborhood identities and linear-programming aggregation also have a long history in Ramsey computation [2, 4]. Gauthier and Brown have since given a kernel-checked HOL4 proof of $R(4,5)=25$ using verified gluing lemmas [6].

The defensible candidate contribution is more specific: we formulate the exact minimum-miss profile $q_s(A)$, its aggregate use across graph-indexed columns, and the equality-rigidity argument that closes the $(85,128)$ endpoint. A focused primary-source search found no earlier statement of this profile inequality or endpoint theorem. That search cannot establish worldwide priority; unindexed code, unpublished filters, or differently phrased antecedents may exist.

No claim is made that the abstract double count is itself new, that this is the first use of hypergraph transversals in Ramsey computation, or that the catalog-conditional endpoint theorem changes $R(5,5)$. The remaining regular degree-18 layers and five other global branches remain open. A meaningful continuation would require a weighted or correlated multi-column capacity inequality with whole-layer reach, a global counting theorem forcing a vertex into this endpoint, or a comparably strong new construction.

A Exceptional endpoint data

For independent reproduction of the hand-checkable terminal step, the exceptional order-18 record is catalog line 51 (zero-based index 50):

```
QznZZYVk{mHZuJuD@\XfQOs}_lo
```

Its unique size-six minimizer is

$$X^* = \{2,3,4,9,11,12\}$$

in zero-based vertex numbering. The ten independent triples missed by X^* are

$$\{0,6,7\},\ \{0,14,15\},\ \{1,8,10\},\ \{1,14,15\},\ \{5,14,15\},$$
$$\{5,16,17\},\ \{6,7,13\},\ \{6,7,17\},\ \{8,10,13\},\ \{8,10,16\}.$$

These labeled data are not additional hypotheses: the separately implemented checker reconstructs them from the pinned graph6 record. The catalog files are attributed under the publisher's stated CC BY 4.0 data license.

AI-assistance and authorship disclosure

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