A four-parameter family of reversible three-species mass-action continua without a common factor

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Abstract

We exhibit a four-dimensional exact rate family on a fixed reversible three-species reaction graph with one linkage class, ten complexes, and ten reversible pairs. Every positive member of the family has full

stoichiometric rank and vanishes on the same compact positive algebraic ellipse. The positive part of rate space is a rational open polyhedral cone, and a nonempty Zariski-open subset of the family has geometrically coprime coordinate polynomials. Thus the equilibrium continuum is generically a height-two component within this constrained family, not a common-factor hypersurface.

We give a primitive positive integer specialization with largest rate 10296 and rate sum 52464. Within the fixed support and conic-preserving family, it simultaneously minimizes both quantities among positive integral rate vectors. Its steady ideal is radical: over $\mathbb{Q}$ its two minimal primes are the conic prime and a degree-fifteen maximal ideal. We also retain the original frozen integer specialization and determine its transverse linear stability along the ellipse. The ellipse has one normally attracting arc and one saddle-type arc, separated by two exactly isolated transition points. All algebraic, integer-optimality, and Sturm claims are accompanied by exact symbolic verifiers.

1. Introduction

For a finite reaction network with complexes $y \in \mathbb{Z}^n_{\geq 0}$, positive rates $\kappa_{y \to y'}$, and concentrations $x \in \mathbb{R}^n_{>0}$, the mass-action vector field is

$$F(x) = \sum_{y \to y'} \kappa_{y \to y'} x^y (y' - y). \tag{1}$$

A network is weakly reversible if every directed reaction belongs to a directed cycle, and reversible if every reaction is paired with its reverse. The stoichiometric subspace is

$$S = \operatorname{span}_{\mathbb{R}}\{y' - y : y \to y'\},$$

and the positive compatibility class through x_0 is $(x_0 + S) \cap \mathbb{R}^n_{>0}$.

Boros, Craciun, and Yu asked whether a weakly reversible mass-action system can have infinitely many positive steady states without a polynomial common factor in the right-hand side [BCY20]. The frozen first version of this work answered that question with a reversible, one-linkage, three-species example whose coordinate gcd is one and whose steady variety contains a positive conic. The present version keeps that theorem unchanged and develops the larger exact structure surrounding it.

The additions are fourfold.

1. We determine all rate vectors on the fixed directed support for which the same conic remains an equilibrium component.
2. We identify the strict positive rate cone and prove that geometric coprimality holds on a nonempty Zariski-open subset.
3. We give a substantially smaller primitive integer specialization and an exact fixed-support optimality certificate.
4. For the frozen first specialization, we classify the two transverse eigenvalues everywhere on the equilibrium ellipse.

No claim of global minimality in complexes or reactions is made. The integer optimality result is explicitly restricted to this fixed support and this conic-preserving rate family.

2. Fixed support and conic

2.1 Complexes and reversible graph

Use species X, Y, Z, with concentration variables x, y, z, and index the complexes as follows.

index	exponent vector	complex
0	$(0, 0, 0)$	0
1	$(0, 0, 1)$	Z
2	$(0, 0, 3)$	$3Z$
3	$(0, 1, 1)$	$Y + Z$
4	$(0, 3, 0)$	$3Y$
5	$(1, 0, 1)$	$X + Z$
6	$(1, 1, 0)$	$X + Y$
7	$(1, 1, 1)$	$X + Y + Z$
8	$(2, 1, 0)$	$2X + Y$
9	$(3, 0, 0)$	$3X$

The ten undirected edges are

$$01, 04, 06, 17, 24, 27, 29, 34, 59, 89, \tag{2}$$

and both orientations of every edge are reactions. The complete rate tables for the two integer specializations appear in Appendix A.

The graph is connected: from 0 one reaches $1, 4, 6$, then $7, 2, 9$, and finally $3, 5, 8$. Hence there is one linkage class. The three reaction differences

$$0 \to Z = (0, 0, 1), \qquad 0 \to 3Y = (0, 3, 0), \qquad 0 \to X + Y = (1, 1, 0)$$

have determinant -3. Therefore the stoichiometric subspace is

$$S = \mathbb{R}^3, \tag{3}$$

independently of the positive rate values. There are no conservation laws, and the unique positive compatibility class is the full positive orthant.

2.2 The common equilibrium conic

Define

$$L = z - x - y + 1 \tag{4}$$

and

$$Q = 7x^2 - 2xy - 16x + 7y^2 - 16y + 16. \tag{5}$$

The ideal

$$\mathfrak{p} = (L, Q) \subset \mathbb{Q}[x, y, z] \tag{6}$$

is a height-two prime. Indeed, eliminating z leaves a nonsingular plane conic. The symmetric matrix of its homogenization is

$$\begin{pmatrix} 7 & -1 & -8 \\ -1 & 7 & -8 \\ -8 & -8 & 16 \end{pmatrix},$$

with determinant -256, so the projective conic is nonsingular and geometrically irreducible.

An exact rational parametrization is

$$\begin{aligned} d(t) &= t^2 - t + 1, \\ x(t) &= \frac{t^2+3}{2d(t)}, \\ y(t) &= \frac{3t^2+1}{2d(t)}, \\ z(t) &= \frac{t^2+t+1}{d(t)}. \end{aligned} \tag{7}$$

Direct substitution gives $L = Q = 0$. All coordinates are positive for every real t, since

$$d(t) = \left(t - \frac{1}{2}\right)^2 + \frac{3}{4}, \qquad t^2 + t + 1 = \left(t + \frac{1}{2}\right)^2 + \frac{3}{4},$$

and the other numerators are $t^2 + 3$ and $3t^2 + 1$. On $(-1, 1)$,

$$z'(t) = \frac{2(1-t^2)}{d(t)^2} > 0,$$

so the parametrized equilibria there are pairwise distinct.

The entire real conic is a compact positive ellipse. On $L = 0$, set $s_0 = x + y = z + 1$ and $u = x - y$. Equation $Q = 0$ becomes

$$4(u^2 + (z-1)^2) = (z+1)^2. \tag{8}$$

It follows that $1/3 \le z \le 3$ and $|u| \le (z+1)/2$, which gives $x, y, z > 0$.

3. The exact four-parameter rate family

Order the twenty directed rates as

$$\begin{aligned} &k_{01}, k_{10}, k_{04}, k_{40}, k_{06}, k_{60}, k_{17}, k_{71}, k_{24}, k_{42}, \\ &k_{27}, k_{72}, k_{29}, k_{92}, k_{34}, k_{43}, k_{59}, k_{95}, k_{89}, k_{98}. \end{aligned} \tag{9}$$

Here the subscripts are complex indices, not species indices.

Theorem 3.1 (complete conic-preserving family). Let $F(k)$ be the mass-action field on the support (2). All three coordinates of $F(k)$ belong to $\mathfrak{p} = (L, Q)$ if and only if, for some $a, b, c, d \in \mathbb{Q}$, the rates in the order (9) are

$$
\begin{aligned}
k_{01} &= (62d - 15c)/48, & k_{10} &= 33(58d - 45c)/160,\\
k_{04} &= 16d/15, & k_{40} &= (c - b)/3,\\
k_{06} &= (154d - 221c)/224, & k_{60} &= (9856d - 3315c)/1470,\\
k_{17} &= 45(154d - 221c)/3136, & k_{71} &= 5c/14,\\
k_{24} &= (15a + 16d)/15, & k_{42} &= (62d - 15b - 15c)/45,\\
k_{27} &= (154d - 192a - 221c)/64, & k_{72} &= 11(58d - 45c)/60,\\
k_{29} &= a, & k_{92} &= 2(31d - 15c)/45,\\
k_{34} &= 88d/15, & k_{43} &= b,\\
k_{59} &= 88d/15, & k_{95} &= c,\\
k_{89} &= 33d/5, & k_{98} &= d.
\end{aligned}
\tag{10}
$$

The conic-preserving rate space has dimension four, and $(a, b, c, d) = (k_{29}, k_{43}, k_{95}, k_{98})$ are free coordinates.

Proof. Use lexicographic order $z > y > x$. The reduced Gröbner basis of $\mathfrak{p}$ is

$$
z - x - y + 1, \qquad y^2 - \frac{2}{7}xy + x^2 - \frac{16}{7}(x + y) + \frac{16}{7}. \tag{11}
$$

Every degree-at-most-three normal form has a unique coefficient vector in the ordered monomial list

$$
(1, x, y, x^2, xy, x^3, x^2y).
$$

Reduce the contribution of each unit directed rate modulo (11), and stack these seven coefficients for the three field coordinates. This constructs a canonical rational matrix $M \in \mathbb{Q}^{21\times 20}$ such that

$$
F_i(k) \in (L, Q) \quad (i = 1, 2, 3) \quad \iff \quad Mk = 0. \tag{12}
$$

The 16×16 minor with zero-based rows

$$
0, 1, 2, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 18, 19
$$

and columns

$$
0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16, 18
$$

has determinant

$$
\frac{7255941120}{823543} = \frac{2^{13}3^{11}5}{7^7} \neq 0. \tag{13}
$$

Thus $\operatorname{rank} M \geq 16$. Substitution of (10) gives four linearly independent kernel vectors, since their entries in positions $12, 15, 17, 19$ form the identity matrix. Hence $\operatorname{rank} M = 16$, the nullity is four, and (10) is exhaustive. The complete matrix and an integer-vector kernel basis are included with the exact family verifier. □

Corollary 3.2 (strict positive cone). All twenty rates in (10) are positive exactly when

$$
a > 0, \quad b > 0, \quad c > 0, \quad d > 0, \quad b < c, \quad 192a + 221c < 154d. \tag{14}
$$

The cone is rationally parametrized by an open positive orthant via

$$c = b + h, \qquad d = \frac{192a + 221(b+h) + s}{154}, \qquad a, b, h, s > 0. \tag{15}$$

Proof. Necessity follows from the four free rates and the two entries

$$k_{40} = \frac{c-b}{3}, \qquad k_{27} = \frac{154d - 192a - 221c}{64}.$$

Conversely, substitute (15) into (10). Every rate becomes a nonzero linear form in a, b, h, s with nonnegative rational coefficients. This proves sufficiency. The inverse coordinates are $h = c - b$ and $s = 154d - 192a - 221c$, so (15) is a bijection. □

Every rate vector in (14) retains the connected reversible support and full stoichiometric rank. Its unique positive compatibility class contains the entire ellipse (6).

4. Generic geometric coprimality

The conic constraint is linear in rate space, but it does not generically force a coordinate common factor.

Lemma 4.1 (scalar extension). Let $f_1, \dots, f_r \in \mathbb{Q}[x_1, \dots, x_n]$. If they have a nonconstant common factor over $\mathbb{C}$, then they have a nonconstant common factor over $\mathbb{Q}$.

Proof. Choose a normalized irreducible complex common factor. Its coefficients are algebraic over $\mathbb{Q}$, because it is a factor of a rational polynomial. Rationality of every f_i forces every distinct Galois conjugate of the chosen factor to divide every f_i. Their product, after multiplication by a scalar, is a nonconstant rational polynomial dividing every f_i. □

Theorem 4.2 (generic geometric gcd one). There is a nonempty Zariski-open subset of the four-dimensional rate family (10) on which

$$\gcd(F_1, F_2, F_3) = 1$$

over $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$.

Proof. Homogenize all three coordinates to degree three in variables x, y, z, w. Let V_j denote the vector space of degree-j forms in these variables. For $e = 1, 2, 3$, triples with a common homogeneous factor of degree e form the image

$$\Sigma_e = \operatorname{im}\left(\mathbb{P}(V_e) \times \mathbb{P}(V_{3-e}^{\oplus 3}) \longrightarrow \mathbb{P}(V_3^{\oplus 3})\right) \tag{16}$$

of the multiplication morphism. The source is projective, so Σ_e is closed. Its affine cone $\widehat{\Sigma}_e \subset V_3^{\oplus 3}$, including the zero triple, is closed. The finite union of these cones pulls back along the linear rate map $\mathbb{A}^4 \to V_3^{\oplus 3}$ to a closed subset of parameter space. This formulation explicitly includes the origin, at which projectivization is undefined.

Every affine common factor homogenizes to a homogeneous common factor. The clean integer specialization in Section 5 has both affine and homogenized coordinate gcd equal to one over $\mathbb{Q}$, as checked by exact factorization. Lemma 4.1 excludes a factor after scalar extension. Hence this specialization lies outside the pulled-back closed set, whose complement is the required nonempty Zariski-open set. □

The theorem is deliberately a generic statement *inside the constrained four-dimensional family*. It does not assert persistence of a continuum under arbitrary perturbations of all rates: leaving (10) generally destroys the conic component, consistently with generic-finiteness results such as [FHP26].

For every parameter outside the closed common-factor locus, the steady ideal is nonzero, is contained in the height-two prime (L, Q), and is contained in no height-one prime. It therefore has height exactly two, and

(L, Q) is a minimal prime over it. Thus the same ellipse is genuinely an irreducible steady-state component throughout this Zariski-open subset, not merely at the two integer specializations below.

5. A clean fixed-support-optimal integer specialization

Set

$$(a, b, c, d) = (653, 1, 70, 915). \tag{17}$$

Its positive-orthant coordinates from (15) are

$$(a, b, h, s) = (653, 1, 69, 64), \tag{18}$$

which provide a compact strict-interiority certificate.

Theorem 5.1 (clean integer construction). The rates obtained from (17), listed in Appendix A, are primitive positive integers. They define a reversible, one-linkage mass-action system with ten complexes, twenty directed reactions, and $S = \mathbb{R}^3$. Its unique positive compatibility class contains the entire ellipse $\mathcal{C} = V(L, Q)$, but its coordinate polynomials have gcd one over $\mathbb{Q}[x, y, z]$, and therefore also over $\mathbb{R}$ and $\mathbb{C}$.

The vector field is

$$\begin{aligned} F_1^{\mathrm{c}} = &-4697x^3 + 6039x^2y - 9177xyz - 5977xy + 10736xz \\ &+ 1960z^3 + 1800z + 560, \\ F_2^{\mathrm{c}} = &\ 915x^3 - 6039x^2y - 9177xyz - 5977xy - 3782y^3 \\ &+ 10736yz + 4888z^3 + 1800z + 3488, \\ F_3^{\mathrm{c}} = &\ 3712x^3 + 18304xyz - 5368xz + 3712y^3 - 5368yz \\ &- 6848z^3 - 10296z + 1160. \end{aligned} \tag{19}$$

Its steady ideal is radical of dimension one. Over $\mathbb{Q}$ it has exactly two minimal primes: the conic prime $\mathfrak{p} = (L, Q)$ and a degree-fifteen maximal ideal disjoint from $\mathfrak{p}$. Over the algebraic closure, the latter component consists of fifteen reduced isolated points.

Proof. Formula (10) and (17) give the twenty rates exactly. Positivity also follows directly from (18). The graph and rank statements were proved in Section 2, and Theorem 3.1 gives $F_i^{\mathrm{c}} \in (L, Q)$. Exact expansion of (1) gives (19). Exact multivariate gcd computation gives

$$\gcd(F_1^{\mathrm{c}}, F_2^{\mathrm{c}}) = \gcd(F_1^{\mathrm{c}}, F_3^{\mathrm{c}}) = \gcd(F_2^{\mathrm{c}}, F_3^{\mathrm{c}}) = 1. \tag{20}$$

Lemma 4.1 upgrades this to geometric coprimality.

For the radical statement, put

$$D = y^2 - yz - y + \frac{7}{16}z^2 - \frac{1}{8}z + \frac{7}{16}, \tag{21}$$

so $\mathfrak{p} = (L, D)$. The exact reduced lexicographic Gröbner basis of $K_{\mathrm{c}} = (F_1^{\mathrm{c}}, F_2^{\mathrm{c}}, F_3^{\mathrm{c}})$, for $x > y > z$, has the form

$$G_0, \qquad DH, \qquad DR, \tag{22}$$

where G_0 is monic linear in x, H is linear in y with nonzero constant leading coefficient, and $R \in \mathbb{Q}[z]$ is irreducible of degree fifteen. Let $\mathfrak{q} = (G_0, H, R)$. Its reduced basis is triangular,

$$x + r_x(z), \qquad y + r_y(z), \qquad R(z),$$

so $\mathfrak{q}$ is maximal of degree fifteen. Exact reduction gives $D \notin \mathfrak{q}$, hence $\mathfrak{p} + \mathfrak{q} = (1)$, and every product of a generator of $\mathfrak{p}$ with a generator of $\mathfrak{q}$ reduces to zero modulo (22). Conversely, each generator of K_c lies in both primes. Therefore

$$K_c = \mathfrak{p}\mathfrak{q} = \mathfrak{p} \cap \mathfrak{q} = \sqrt{K_c}. \tag{23}$$

Characteristic zero makes R separable, yielding fifteen reduced points after scalar extension. Every reduction and factorization in this argument is replayed by the exact verifier. □

5.1 Fixed-support integer optimality

The clean vector has

$$\max_j k_j = 10296, \qquad \sum_j k_j = 52464, \qquad \gcd(k_0, \dots, k_{19}) = 1. \tag{24}$$

Proposition 5.2 (bounded exact optimum). Among all positive integral vectors in the fixed-support family (10), the clean vector simultaneously attains the smallest possible largest entry and the smallest possible sum of entries.

Proof. Integrality of $k_{04} = 16d/15$ and $k_{71} = 5c/14$ forces

$$15 \mid d, \qquad 14 \mid c. \tag{25}$$

If $\max_j k_j < 10296$, then $k_{34} = 88d/15$ gives $d < 1755$. If $\sum_j k_j < 52464$, then

$$k_{34} + k_{59} + k_{89} + k_{98} = \frac{58}{3}d$$

gives $d < 2714$. These bounds reduce both strict-improvement questions to finite exact searches.

For each multiple d of 15 in the relevant range and each multiple c of 14, positivity permits the safe bounds $c < d$, $1 \le b < c$, and $1 \le a < d$. The formulas (10) separate into entries depending only on (c, d), on (a, c, d), and on (b, c, d). The verifier enumerates these finite integer sets using rational arithmetic, rejects every nonintegral or nonpositive entry, and computes the exact minimum of the maximum and of the sum. It returns respectively 10296 and 52464, both attained at (17). No floating-point optimization or rational reconstruction enters the certificate. □

This proposition is not a network-wide minimality theorem. It fixes the twenty directed reactions, the conic, and primitive integer normalization. A common positive scaling of all rates merely rescales time.

6. The frozen first specialization

The original theorem remains valid without alteration. Its free parameters are

$$(a, b, c, d) = (3920, 3920, 15680, 658560) = 3920(1, 1, 4, 168), \tag{26}$$

and its positive-orthant coordinates are

$$(a, b, h, s) = (3920, 3920, 11760, 97200320). \tag{27}$$

Thus the frozen rate vector is also a strict interior point of the exact family, not a boundary specialization.

Theorem 6.1 (frozen v1 construction). The original rates in Appendix A define a reversible, one-linkage, three-species mass-action system with $S = \mathbb{R}^3$, the full positive ellipse $\mathcal{C}$ as an equilibrium component, and

$$\gcd(F_1^0, F_2^0, F_3^0) = 1.$$

Its steady ideal is radical and equals the intersection of the conic prime with a disjoint degree-fifteen maximal ideal over $\mathbb{Q}$.

Its vector field is

$$\begin{aligned} F_1^0 = &-3380608x^3 + 4346496x^2y - 6878928xyz - 4380128xy \\ &+ 7727104xz + 1530515z^3 + 1405575z + 437290, \\ F_2^0 = &\, 658560x^3 - 4346496x^2y - 6878928xyz - 4380128xy \\ &- 2722048y^3 + 7727104yz + 3637907z^3 + 1405575z + 2544682, \\ F_3^0 = &\, 2706368x^3 + 13746656xyz - 3863552xz + 2706368y^3 \\ &- 3863552yz - 5168422z^3 - 7732494z + 845740. \end{aligned} \tag{28}$$

Proof. Equations (26) and (10) reproduce the frozen rates exactly, and Theorem 3.1 proves conic containment. Exact factorization makes the three primitive coordinate cubics irreducible and pairwise nonassociate, hence pairwise coprime. The radical decomposition follows by the same triangular Gröbner-basis certificate used in (21)–(23). In the frozen case, the degree-fifteen eliminant is independently certified irreducible by its reduction modulo the prime 19. The exact v1 verifier reconstructs every claim from the directed reaction table. ☐

At $x_0 = (3/2, 1/2, 1)$, the compatibility class is all of $\mathbb{R}^3_{>0}$, and (7) on $(-1, 1)$ gives infinitely many distinct positive equilibria in that class. The continuum is genuinely height two: all coordinates lie in (L, Q), but their gcd is one.

7. Transverse stability of the frozen ellipse

This section concerns F^0, not the clean specialization. Let $J(t) = DF^0(x(t), y(t), z(t))$. Differentiating $F^0(x(t), y(t), z(t)) = 0$ shows that the tangent direction is a zero eigenvector. Write the other two eigenvalues as $\lambda_1(t), \lambda_2(t)$.

Theorem 7.1 (exact transverse classification). There are exactly two real numbers

$$\alpha \in (-4, -3), \qquad \beta \in (9/10, 1) \tag{29}$$

at which normal hyperbolicity is lost. The finite-t portion of the ellipse is normally attracting for

$$\alpha < t < \beta,$$

and is transversely saddle-type for $t < \alpha$ or $t > \beta$. At each of α, β, one transverse eigenvalue is zero and the other is strictly negative. The point omitted by the affine parameter $t \in \mathbb{R}$, namely the limit $t = \infty$, is also saddle-type.

Proof. Exact characteristic-polynomial reduction gives

$$\det(\lambda I - J(t)) = \lambda(\lambda^2 - \tau(t)\lambda + \pi(t)), \tag{30}$$

where

$$\tau(t) = -\frac{8T(t)}{d(t)^2}, \qquad \pi(t) = -\frac{6272N(t)}{d(t)^4}, \tag{31}$$

with

$$T(t) = 5399367t^4 + 1602005t^3 + 11579010t^2 + 1602005t + 6979911 \tag{32}$$

and

$$\begin{aligned} N(t) = {} & 5730530769t^8 + 20026244073t^7 + 29613209084t^6 \\ & + 118245415239t^5 - 38238695578t^4 + 127692520263t^3 \\ & - 127590858244t^2 + 10579139049t - 79465564719. \end{aligned} \tag{33}$$

The transverse discriminant is

$$(\lambda_1 - \lambda_2)^2 = \frac{64E(t)}{d(t)^4}, \tag{34}$$

where

$$\begin{aligned} E(t) = {} & 31399532062137t^8 + 25149913538286t^7 + 139213446954293t^6 \\ & + 100751092465458t^5 + 199590946186248t^4 + 109518436416306t^3 \\ & + 114191722124597t^2 + 26510727150318t + 17568656198073. \end{aligned} \tag{35}$$

Exact Sturm counts give no real roots for T or E; evaluation at zero shows both are positive on $\mathbb{R}$. Thus the trace is always negative and the two transverse eigenvalues are real and distinct.

The same exact Sturm procedure gives precisely two real roots of N, one in each interval (29) and none elsewhere. Define these roots to be α, β. The sign checks

$$N(-4) > 0, \quad N(-3) < 0, \quad N(9/10) < 0, \quad N(1) > 0$$

show that $N < 0$ between the roots and $N > 0$ outside. Hence $\pi > 0$ on (α, β) and $\pi < 0$ outside. Negative trace, positive product, and real roots give two negative transverse eigenvalues on the inner interval; negative product gives one eigenvalue of each sign on the outer intervals. At a root of N, equation (30) has transverse roots 0 and $\tau < 0$. Finally, the leading coefficients in (31)–(35) give $\pi(\infty) < 0$, proving saddle type at the omitted point. □

For orientation only,

$$\alpha \approx -3.8135049145, \qquad \beta \approx 0.9130496953.$$

These decimals play no role in the proof. On the interval $(-1, 1)$ used to display the v1 continuum, the normally attracting part is $(-1, \beta)$, the point β is nonhyperbolic, and $(\beta, 1)$ is saddle-type.

8. Why the mechanism is genuinely height two

For every rate vector in the family,

$$F_i \in (L, Q).$$

For the two exact specializations, no nonconstant polynomial divides all three coordinates. Consequently their equilibrium continuum is not obtained by multiplying a smaller vector field by a common scalar polynomial. In addition:

- the stoichiometric class is three-dimensional, not a plane on which the field vanishes identically;
- all species occur, and reaction differences span all three directions;
- the coordinate fields are not artificial duplicates;
- the vector fields are nonzero off the conic; and
- the Jacobian has rank two at $(3/2, 1/2, 1)$ for both specializations.

Since (L, Q) is a height-two prime and the coordinate gcd is one, the steady ideal has height exactly two. The equilibrium conic is therefore an actual irreducible component, not merely a sampled subset of a hidden hypersurface.

9. Minimality and structural obstructions

9.1 Three species are necessary

No one- or two-species system can have a positive-dimensional equilibrium continuum in one compatibility class while retaining coordinate gcd one. In one species, a positive-dimensional zero set forces the single coordinate polynomial to vanish identically. In two species:

- if $\dim S = 1$, all reaction vectors are collinear, so $F = vf(x, y)$ for a fixed vector v and scalar polynomial f;
- if $\dim S = 2$, a positive-dimensional common zero set gives a height-one prime over the steady ideal, and factoriality of $\mathbb{Q}[x, y]$ supplies a nonconstant common divisor.

Thus three species are globally minimal for the stated phenomenon.

9.2 One linkage class forces full rank in three species

Suppose a three-species, one-linkage network has $\dim S = 2$. Let a primitive integer vector w span $S^\perp$. Connectivity makes $w \cdot y = m$ constant over all complexes. The torus action

$$T_\rho(x_1, x_2, x_3) = (\rho^{w_1} x_1, \rho^{w_2} x_2, \rho^{w_3} x_3)$$

satisfies $F_i(T_\rho x) = \rho^m F_i(x)$. Its infinitesimal direction at a positive point p has inner product

$$\sum_i w_i^2 p_i > 0$$

with the compatibility-class normal w, so it is transverse to the class. A one-dimensional equilibrium stratum in one class therefore sweeps out a two-dimensional equilibrium set. The steady ideal then has height at most one (unless the whole field is zero), and factoriality of $\mathbb{Q}[x, y, z]$ forces a common irreducible divisor. Rank one is already excluded by collinearity. Hence full rank is necessary under the one-linkage hypothesis.

9.3 Deficiency and a global five-complex bound

Write m for the number of active complexes, ℓ for the number of linkage classes, $s = \dim S$, and $\delta = m - \ell - s$. Every weakly reversible target system has $\delta \geq 1$. Here is a direct proof. Let Y be the complex matrix, let A_κ be the kinetic Laplacian, and let H be the subspace of vectors whose coordinates sum to zero separately on each linkage class. Then $Y(H) = S$ and

$$\delta = \dim \ker(Y|_H).$$

If $\delta = 0$, the equilibrium identity $YA_\kappa\Psi(x) = 0$ and $A_\kappa\Psi(x) \in H$ imply $A_\kappa\Psi(x) = 0$: every positive equilibrium is complex-balanced. Weak reversibility makes the kernel of each linkage Laplacian one-dimensional and positive. Hence two positive equilibria x, x' have proportional monomial vectors on each linkage class, giving

$$\log x - \log x' \in S^\perp.$$

If they lie in one compatibility class, then $x - x' \in S$, and strict monotonicity of the logarithm yields

$$0 = (x - x') \cdot (\log x - \log x') = \sum_i (x_i - x_i')(\log x_i - \log x_i'),$$

whose summands are nonnegative and vanish only when $x_i = x_i'$. Thus a deficiency-zero weakly reversible system has at most one positive equilibrium per class and cannot have the target continuum.

Now $s \geq 2$ by Section 9.1 and $\delta \geq 1$, so $m \geq 4$. Equality would force $(m, \ell, s, \delta) = (4, 1, 2, 1)$, which Section 9.2 excludes. Therefore every three-species weakly reversible realization of the target, even with multiple linkage classes, has at least five complexes. Under one linkage it must also have $s = 3$ and $\delta = m - 4 \geq 1$.

The construction uses ten complexes, ten reversible pairs, maximum complex degree three, and deficiency

$$10 - 1 - 3 = 6.$$

No claim that these complex, reaction, or deficiency counts are globally minimal is made.

10. Relation to earlier work and scope

The construction and exact verifier were completed before the targeted literature audit. Boros, Craciun, and Yu constructed weakly reversible systems with continua of positive equilibria and explicitly asked whether such a continuum can occur without a coordinate common factor [BCY20]. Their examples use a common-factor hypersurface. Their displayed parametrization also contains a positive-dimensional fixed-support family preserving the same curve, so rate flexibility by itself is not new; every member of that family retains the displayed common scalar factor. The systems here answer their question by a height-two prime component and add a fixed-support family that is geometrically gcd one on a nonempty Zariski-open subset.

Feliu, Henriksson, and Pascual-Escudero prove generic-finiteness results for steady-state varieties and revisit fine-tuned common-factor examples [FHP26]. Those results concern generic changes in ambient rate and total parameters. They do not preclude the exact codimension-sixteen linear family inside the twenty-dimensional fixed-support rate space found here.

A narrow primary-source audit through 1 August 2026 found no earlier weakly reversible fixed-support positive rate family preserving a continuum in one compatibility class while being generically free of a coordinate common factor within that family. Thus, to our knowledge, this is the first explicit example with that conjunction. The four-dimensional family has codimension sixteen in ambient rate space, so the statement is relative rather than a claim of robustness under arbitrary rate perturbations. This is a conservative audit conclusion, not an exhaustive universal priority claim.

11. Exact verification

Four exact replay layers support the claims.

1. The frozen verifier reconstructs the original network, conic identities, gcd, graph, stoichiometry, and radical decomposition.
2. The family verifier reconstructs the canonical 21×20 remainder matrix, rank minor, integer-vector kernel basis, positive cone, and homogenized gcd-one witness.
3. The strengthening verifier reconstructs the clean field and radical decomposition, performs the bounded integer enumeration, derives the characteristic polynomial, and executes all Sturm counts.
4. A separate 993-line v2 audit independently reconstructs the family matrix, both rate specializations, both radical decompositions, the two integer optima, and hand-built Sturm chains, then checks a frozen machine-readable result file.

Appendix B gives one command that runs all four layers and additional cross-checks. All polynomial identities use exact rational arithmetic. The decimal approximations in Section 7 are explicitly non-evidentiary.

Generative-AI systems were used substantively in discovery, exact computation, adversarial review, verifier development, and manuscript preparation. The named human author accepts responsibility for the released claims and artifacts. Neither exact replay nor public timestamping is a substitute for independent expert review.

References

[**BCY20**] B. Boros, G. Craciun, and P. Y. Yu, "Weakly Reversible Mass-Action Systems With Infinitely Many Positive Steady States," *SIAM Journal on Applied Mathematics* 80 (2020), 1936–1946. doi:10.1137/19M1303034.

[**FHP26**] E. Feliu, O. Henriksson, and B. Pascual-Escudero, "The Generic Geometry of Steady State Varieties," *SIAM Journal on Applied Algebra and Geometry* 10 (2026), 519–548. doi:10.1137/25M1731289.

Appendix A. Complete directed rate tables

The rows below use the directed order (9). The v1 and clean columns are two distinct exact members of the same family; they are not combined in one system.

j	directed reaction	frozen v1 rate	clean rate
0	$0 \to 1$	845740	1160
1	$1 \to 0$	7732494	10296
2	$0 \to 4$	702464	976
3	$4 \to 0$	3920	23
4	$0 \to 6$	437290	560
5	$6 \to 0$	4380128	5977
6	$1 \to 7$	1405575	1800
7	$7 \to 1$	5600	25
8	$2 \to 4$	706384	1629
9	$4 \to 2$	900816	1237
10	$2 \to 7$	1518755	1
11	$7 \to 2$	6873328	9152
12	$2 \to 9$	3920	653
13	$9 \to 2$	896896	1214
14	$3 \to 4$	3863552	5368
15	$4 \to 3$	3920	1
16	$5 \to 9$	3863552	5368

j	directed reaction	frozen v1 rate	clean rate
17	$9 \to 5$	15680	70
18	$8 \to 9$	4346496	6039
19	$9 \to 8$	658560	915

The clean vector is primitive because it contains entries equal to one. Its maximum and sum are 10296 and 52464. The frozen vector has maximum 7732494 and sum 39165070.

Appendix B. Reproduction commands

In the released archive, run

```
./reproduce.sh
```

to create the locked local environment and replay every exact layer. From a repository checkout with the dependencies already installed, the combined wrapper is

```
.venv/bin/python weakly_reversible_continuum_no_common_factor/manuscript_v2_draft/verify_v2_claims.py
```

to replay the v1, family, clean-rate, integer-optimality, radical, and stability checks. To compile this manuscript to PDF, run

```
sh weakly_reversible_continuum_no_common_factor/manuscript_v2_draft/build_pdf.sh
```

The build command only renders the manuscript; it does not create a release, DOI, or external publication.