

A 14-variable polynomial map with everywhere unipotent Jacobian and a three-point fiber

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Abstract

We give an exact rational polynomial vector $g \in \mathbb{Q}[Z]^{14}$, with 24 monomials, such that

$$\det(I + sJg) = 1$$

identically, and such that $I + g$ has three displayed rational points in one fiber. Generically, Jg is regular nilpotent, with Jordan type (14). The generic degree is three and the geometric monodromy is S_3 . For the image operator $E(\xi^\alpha p(Z)) = \partial_Z^\alpha p(Z)$, put $A = -\sum_{j=1}^{14} \xi_j g_j$ and $b = x + y + u_{11}$. We prove

$$E(A^m) = 0, \quad E(bA^m) \neq 0 \quad (m \geq 1).$$

Thus SIC(14) fails, with an obstruction at every exponent. We also record degree-seven homogeneous and Hessian-nilpotent companions. The dimension and term counts are proved optimal only in a precisely stated constant-state realization ansatz.

Verification disclaimer. The human author is a complete amateur exploring the limits of AI-assisted mathematics and cannot independently verify these claims. This is a research artifact for expert checking, not an established result. Priority and minimality have only the narrow scope stated below.

1 The three-variable source

Put $u = 1 + xy$ and

$$\begin{aligned} F_1 &= u^3 z + y^2 u(4 + 3xy), \\ F_2 &= y + 3xu^2 z + 3xy^2(4 + 3xy), \\ F_3 &= 2x - 3x^2 y - x^3 z. \end{aligned}$$

Use the identity-linear normalization

$$\Phi = (F_3/2, F_2, F_1) = X + H_2 + H_3 + \cdots + H_7. \tag{1}$$

The nonzero homogeneous pieces are displayed in Table 1.

Direct differentiation gives $\det J\Phi = 1$. The three points

$$r_0 = (0, 0, -\tfrac{1}{4}), \quad r_+ = (1, -\tfrac{3}{2}, \tfrac{13}{2}), \quad r_- = (-1, \tfrac{3}{2}, \tfrac{13}{2}) \tag{2}$$

all map to $(0, 0, -1/4)$.

d	$H_{d,1}$	$H_{d,2}$	$H_{d,3}$
2	0	$3xz$	$4y^2$
3	$-\frac{3}{2}x^2y$	$12xy^2$	$3xyz$
4	$-\frac{1}{2}x^3z$	$6x^2yz$	$7xy^3$
5	0	$9x^2y^3$	$3x^2y^2z$
6	0	$3x^3y^2z$	$3x^2y^4$
7	0	0	x^3y^3z

Table 1: Homogeneous pieces of $\Phi - I$.

2 The 14-variable construction

Take state coordinates

$$U = (u_{11}, u_{12}; u_{21}, u_{22}, u_{23}, u_{24}; u_{31}, u_{32}, u_{33}, u_{34}, u_{35}).$$

Let $B \in \text{Mat}_{3 \times 11}(\mathbb{Q})$ select the three chain heads, so $BU = (u_{11}, u_{21}, u_{31})^T$. Let $N = J_2 \oplus J_4 \oplus J_5$, with ones immediately above the diagonal in each Jordan block, and put

$$C = \begin{pmatrix} H_{3,1} \\ H_{4,1} \\ H_{3,2} \\ H_{4,2} \\ H_{5,2} \\ H_{6,2} \\ H_{3,3} \\ H_{4,3} \\ H_{5,3} \\ H_{6,3} \\ H_{7,3} \end{pmatrix}.$$

Define

$$g(X, U) = (H_2(X) + BU, -C(X) - NU). \quad (3)$$

In the coordinate order $Z = (x, y, z, U)$, its components are

$$\begin{aligned} g_1 &= u_{11}, & g_2 &= 3xz + u_{21}, & g_3 &= 4y^2 + u_{31}, \\ g_4 &= \frac{3}{2}x^2y - u_{12}, & g_5 &= \frac{1}{2}x^3z, \\ g_6 &= -12xy^2 - u_{22}, & g_7 &= -6x^2yz - u_{23}, \\ g_8 &= -9x^2y^3 - u_{24}, & g_9 &= -3x^3y^2z, \\ g_{10} &= -3xyz - u_{32}, & g_{11} &= -7xy^3 - u_{33}, \\ g_{12} &= -3x^2y^2z - u_{34}, & g_{13} &= -3x^2y^4 - u_{35}, \\ g_{14} &= -x^3y^3z. \end{aligned} \quad (4)$$

There are exactly 24 monomials. Hence

$$A(\xi, Z) = -\sum_{j=1}^{14} \xi_j g_j(Z) \quad (5)$$

has 24 monomials in 28 indeterminates and total degree eight.

3 Determinant pencil and regular nilpotency

Theorem 1. *For the vector in (4),*

$$\det(I + sJg) = 1.$$

The generic Jordan type of Jg is (14). The map $I + g$ has three distinct rational points in one fiber, has generic degree three, and has geometric monodromy S_3 .

Proof. The Jacobian pencil has block form

$$I + sJg = \begin{pmatrix} I_3 + sJH_2 & sB \\ -sJC & I_{11} - sN \end{pmatrix}.$$

Since $N^5 = 0$,

$$\det(I - sN) = 1, \quad (I - sN)^{-1} = \sum_{k=0}^4 s^k N^k.$$

The chain definition gives $BN^kC = H_{k+3}$ for $0 \leq k \leq 4$, with absent components zero. The Schur complement is therefore

$$\begin{aligned} I_3 + sJH_2 + s^2B(I - sN)^{-1}JC &= I_3 + \sum_{d=2}^7 s^{d-1}JH_d(X) \\ &= J\Phi(sX). \end{aligned}$$

Its determinant is one, proving the pencil identity. Cayley–Hamilton gives $(Jg)^{14} = 0$, while exact multiplication gives

$$((Jg)^{13})_{1,5} = -3x^6y^4z \neq 0. \quad (6)$$

The generic nilpotency index is therefore 14, which forces the single Jordan block (14).

For a source point r , put $U(r) = (I - N)^{-1}C(r)$. Then

$$(I + g)(r, U(r)) = (\Phi(r), 0).$$

Thus the three points in (2) lift to the explicit points displayed in (7) below and have common image.

Finally, for a generic target (Y, V) the state equations give

$$U = (I - N)^{-1}(V + C(X)), \quad \Phi(X) = Y - B(I - N)^{-1}V.$$

Thus $I + g$ is stably equivalent to $\Phi \times I_{11}$. The exact cubic resolvent, irreducibility, and nonsquare-discriminant certificates for Φ are given in Discovery 04; they show generic degree three and geometric monodromy S_3 . $\square$

The lifted points promised in the proof are

$$\begin{aligned} p_0 &= (0, 0, -\tfrac{1}{4}; 0, 0; 0, 0, 0, 0; 0, 0, 0, 0, 0), \\ p_+ &= (1, -\tfrac{3}{2}, \tfrac{13}{2}; -1, -\tfrac{13}{4}; -18, -45, \tfrac{27}{2}, \tfrac{351}{8}; \\ &\quad -\tfrac{63}{4}, \tfrac{27}{2}, \tfrac{297}{8}, -\tfrac{27}{4}, -\tfrac{351}{16}), \\ p_- &= (-1, \tfrac{3}{2}, \tfrac{13}{2}; 1, \tfrac{13}{4}; 18, 45, -\tfrac{27}{2}, -\tfrac{351}{8}; \\ &\quad -\tfrac{63}{4}, \tfrac{27}{2}, \tfrac{297}{8}, -\tfrac{27}{4}, -\tfrac{351}{16}). \end{aligned} \quad (7)$$

Their common image is $(0, 0, -1/4, 0, \dots, 0)$.

For every nonzero scalar s , the same elimination applied to $I + sg$ gives

$$s^{-1}\Phi(sX) = Y - sB(I - sN)^{-1}V.$$

Consequently every nonzero pencil member is noninjective and retains generic degree three and monodromy S_3 .

4 Consequences for injectivity conjectures

The derivative of $I + g$ is everywhere unipotent, but (7) proves noninjectivity. This gives an explicit dimension-14 counterexample to Campbell's Unipotent Jacobian Univalence Conjecture and to the all-eigenvalues-one form of Chamberland's injectivity conjecture.

It also gives a concrete failure of Kulikov's $JN(14)$. For $c \in \mathbb{C}^{14}$ set

$$N_c(Z) = c - g(Z).$$

Then $JN_c = -Jg$ is nilpotent, and fixed points of N_c are exactly the points of $(I + g)^{-1}(c)$. The displayed value gives three rational fixed points; a generic value gives three geometric fixed points.

These are algebraic injectivity statements. They should not be confused with the classical dynamical Markus–Yamabe conjecture, which has older counterexamples, or with valid theorems using singular-value rather than eigenvalue bounds.

5 The Special Image Conjecture

For n pairs of variables define

$$E_n : \mathbb{C}[\xi_1, \dots, \xi_n, Z_1, \dots, Z_n] \longrightarrow \mathbb{C}[Z], \quad E_n(\xi^\alpha p) = \partial_Z^\alpha p.$$

The Special Image Conjecture says that $\ker E_n$ is a Mathieu–Zhao subspace.

Lemma 2 (scalar-parameter inversion). *Suppose $\det(I + tJg) = 1$, put $A = -\xi \cdot g$, and let Q_t be the formal inverse of $I + tg$. For every polynomial p ,*

$$p(Q_t(Z)) = \sum_{m \geq 0} \frac{t^m}{m!} E_n(pA^m). \quad (8)$$

Proof. Apply the Abhyankar–Gurjar inversion formula to $I + tg = I - H_t$, where $H_t = -tg$. The determinant factor is one, and the multinomial theorem collects terms of total ξ -degree m . The formula is valid t -adically: each coefficient receives only finitely many contributions. $\square$

Theorem 3. *For A in (5) and $b = x + y + u_{11}$,*

$$E_{14}(A^m) = 0, \quad E_{14}(bA^m) \neq 0 \quad (m \geq 1).$$

Consequently SIC(14) is false.

Proof. Taking $p = 1$ in (8) gives $E_{14}(A^m) = 0$ for every $m \geq 1$. For the sharp obstruction, specialize the inverse target to

$$Z_* = (\tfrac{1}{2}, 0, 1, 0, \dots, 0).$$

Let $\tau \in t\mathbb{Q}[[t]]$ be the unique solution of

$$\tau + t\tau^3 = \frac{t}{2}. \quad (9)$$

State elimination followed by the source cubic reconstruction gives

$$Q_{t,x}(Z_*) = \tau', \quad Q_{t,y}(Z_*) = -3\tau, \quad Q_{t,u_{11}}(Z_*) = -\frac{\tau' - 1/2}{t}.$$

The last identity is also immediate from the first target equation $Q_x + tQ_{u_{11}} = 1/2$.

Write $b(Q_t)(Z_*) = \sum_{m \geq 0} q_m t^m$. Lagrange inversion in (9) gives

$$\begin{aligned} q_{3k} &= (-1)^k \frac{\binom{3k+1}{k}}{2^{2k+1}}, \\ q_{3k+1} &= (-1)^{k+1} \frac{3\binom{3k+1}{k}}{(3k+1)2^{2k+1}}, \\ q_{3k+2} &= (-1)^k \frac{\binom{3k+4}{k+1}}{2^{2k+3}}. \end{aligned} \quad (10)$$

Every number in (10) is nonzero. Comparing coefficients in (8) yields

$$E_{14}(bA^m)(Z_*) = m!q_m \neq 0$$

for every $m \geq 1$. □

6 Optimality inside the state ansatz

No global minimality claim is made. Consider only realizations

$$\tilde{g}(X, U) = (H_2(X) + \tilde{B}U, -\tilde{C}(X) - \tilde{N}U) \quad (11)$$

with constant matrices, nilpotent $\tilde{N}$, and exact Markov parameters

$$\tilde{B}\tilde{N}^k\tilde{C} = H_{k+3} \quad (k \geq 0).$$

Order the eleven distinct tail monomials as

$$x^2y, xy^2, xyz, x^3z, x^2yz, xy^3, x^2y^3, x^2y^2z, x^3y^2z, x^2y^4, x^3y^3z.$$

Let $M_k \in \text{Mat}_{3 \times 11}(\mathbb{Q})$ be the coefficient matrix of H_{k+3} in these columns, with $M_k = 0$ for $k \geq 5$. For $0 \leq j \leq 5$, form $\mathcal{H}_j = (M_{j+r+c})_{0 \leq r, c \leq 5-j}$. Their exact ranks are

$$11, 8, 5, 3, 1, 0.$$

For $\mathcal{H}_0$, in standard block order, rows 1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 15 and columns 1, $\dots$, 11 give the exact minor

$$275562 = 2 \cdot 3^9 \cdot 7 \neq 0.$$

Every realization factors $\mathcal{H}_0$ through its state space, so at least eleven states are required. The shifted matrix $\mathcal{H}_1$ factors through $\tilde{N}$, and hence its rank gives $\text{rank } \tilde{N} \geq 8$. Thus dimension 14 is minimal in (11). Moreover $\text{rank } \tilde{B} \geq 3$, and $\tilde{C}$ must contain all eleven monomial inputs. Together with the two terms of H_2 , this gives

$$2 + 3 + 8 + 11 = 24$$

as a term lower bound. The displayed construction attains it.

7 Homogeneous companions

Introduce w and homogenize to degree seven:

$$h(Z, w) = (w^7 g(Z/w), 0) \in \mathbb{Q}[Z, w]^{15}. \quad (12)$$

This vector has 24 terms, and its zero last row gives

$$\det(I + sJh) = \det(I + sw^6 Jg(Z/w)) = 1.$$

Thus Jh is nilpotent. The points $(p_0, 1), (p_+, 1), (p_-, 1)$ collide under $I + h$, while degree and monodromy remain 3 and S_3 . The exact checker certifies $(Jh)^{13} \neq 0$, so its generic nilpotency index is at least 14. We make no exact index assertion without a separate degree-14 zero certificate.

As a standard consequence of the de Bondt–van den Essen symmetrization lemma, the polynomial

$$P(A, B) = i \sum_{j=1}^{15} h_j(A + iB)B_j \quad (13)$$

is a homogeneous degree-eight Hessian-nilpotent polynomial in 30 variables. The usual linear solve with $I + Jh(x)^T$ transports two displayed colliding points to a collision of $I - \nabla P$. This paragraph invokes the standard lemma; the independently checked headline artifact is the 14-variable map.

8 Scope and priority

The state-space count

$$3 + (4 - 2) + (6 - 2) + (7 - 2) = 14$$

is classical degree bookkeeping in Kulikov’s reduction. The abstract existence of a 14-dimensional reduction is not claimed as new. The asserted contribution is the sparse explicit realization, its regular-nilpotent and three-point certificates, and the consequence cascade, particularly the every-exponent SIC(14) formula.

At the audit cutoff, no earlier public source was located giving this exact object or an explicit SIC(n) witness for $n \leq 14$. That is provisional evidence, not proof of worldwide priority. No global smallest-dimension or smallest-term claim is made.

References

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