

Full wreath-product monodromy through the third iterate of an explicit Keller map

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with heavy assistance from ChatGPT 5.6 Sol

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Abstract

Let $F : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ be the newly announced noninjective polynomial map with constant Jacobian determinant -2 and geometric monodromy S_3 . We determine the geometric monodromy of its second and third iterates:

$$\text{Mon}(F^{\circ 2}) = S_3 \wr S_3, \quad \text{Mon}(F^{\circ 3}) = S_3 \wr S_3 \wr S_3.$$

The respective degrees are 9 and 27, and the group orders are 1296 and 13,060,694,016. On the target line $(1, 2, s)$, exact primitive eliminants have one-edge Newton polygons at infinity, giving a 9-cycle and a 27-cycle. Each discriminant has a squarefree divisor of multiplicity one and hence gives a single deepest-block transposition; at level three, exact leading-coefficient and reconstruction-denominator guards are included. Elementary group lemmas force the full wreath products. Postcomposition by $\text{diag}(1/4, 1, 1)$ gives a noninjective unit-Jacobian map realizing the pair $(9, S_3 \wr S_3)$. We also prove that every iterate $F^{\circ m}$ has a full 3^m -cycle in its geometric inertia at infinity. Exact SymPy, dependency-free finite-field, PARI/GP, and GAP checks accompany the argument.

Verification disclaimer. The human author is a complete amateur exploring the limits of AI-assisted mathematics and cannot independently verify these claims. This is a research artifact for expert checking, not an established result. No guarantee of worldwide priority is made.

1 The map and the result

Put $u = 1 + xy$ and define $F = (F_1, F_2, F_3)$ by

$$\begin{aligned} F_1 &= u^3 z + y^2 u(4 + 3xy), \\ F_2 &= y + 3xu^2 z + 3xy^2(4 + 3xy), \\ F_3 &= 2x - 3x^2 y - x^3 z. \end{aligned} \tag{1}$$

Exact expansion gives $\det JF = -2$. Moreover,

$$p_0 = (0, 0, -\tfrac{1}{4}), \quad p_1 = (1, -\tfrac{3}{2}, \tfrac{13}{2}), \quad p_2 = (-1, \tfrac{3}{2}, \tfrac{13}{2}) \tag{2}$$

all map to $(-1/4, 0, 0)$.

For a dominant polynomial map H , write $\text{Mon}(H)$ for the Galois group of the normal closure of the induced function-field extension, in its natural action on a generic fiber.

Put $W_1 = S_3$ and $W_{m+1} = S_3^{3^m} \rtimes W_m$ in the natural action on the depth- $(m+1)$ ternary tree.

Theorem 1. *The second and third iterates have generic degrees 9 and 27, respectively, and*

$$\text{Mon}(F^{\circ 2}) = W_2 = S_3 \wr S_3 \leq S_9, \quad \text{Mon}(F^{\circ 3}) = W_3 = S_3 \wr S_3 \wr S_3 \leq S_{27}.$$

Moreover, $|W_2| = 6^4 = 1296$ and $|W_3| = 6^{1+3+9} = 13,060,694,016$.

Proposition 2. *For every $m \geq 1$, the geometric monodromy of $F^{\circ m}$ contains a cycle of length 3^m .*

Proposition 3. *Write $G = F^{\circ 2}$ and let $\widehat{G} = (G_1/4, G_2, G_3)$. Then $\det J\widehat{G} = 1$, $\text{Mon}(\widehat{G}) = S_3 \wr S_3$, and $\widehat{G}$ is noninjective. The three points in (2) all map under $\widehat{G}$ to $(0, 0, -1/2)$.*

The last proposition follows immediately from the theorem and $\det J(F \circ F) = 4$.

2 The cubic inverse resolvent

For a target (a, b, c) , introduce

$$C_{a,b,c}(t) = 2at^3 - bt^2 + 2t - c. \quad (3)$$

For a simple root t , the generic preimage is reconstructed by

$$y = -\frac{bt^2 + 3c - 6t}{2t^2}, \quad x = \frac{t}{1 - ty}, \quad z = \frac{2x - 3x^2y - c}{x^3}. \quad (4)$$

Substitution reduces every coordinate of $F(x, y, z) - (a, b, c)$ to zero modulo $C_{a,b,c}(t)$. The parameter is $t = x/(1 + xy)$, reciprocal to the usual parameter $y + 1/x$.

The three roots of (3) parametrize the generic fiber. For $F \circ F$, each of three outer roots supports a second cubic of inner roots. Consequently the generic degree is nine and

$$\text{Mon}(F \circ F) \leq S_3 \wr S_3. \quad (5)$$

One further iteration gives the natural depth-three ternary-tree block system, so $\text{Mon}(F^{\circ 3}) \leq W_3$; its action on the nine bottom blocks is the monodromy of the intermediate $F^{\circ 2}$ cover.

3 A degree-nine slice

Restrict the target to $(a, b, c) = (1, 2, s)$. Let t satisfy

$$2t^3 - 2t^2 + 2t - s = 0$$

and reconstruct $(x(t), y(t), z(t))$ using (4). An inner resolvent root r satisfies

$$2x(t)r^3 - y(t)r^2 + 2r - z(t) = 0.$$

Eliminating t and removing the vertical content $256s^7$ gives the primitive polynomial $P(r, s)$ below.

$$\begin{aligned}
P = & 128r^9s - 256r^8 + (2592s^2 - 3072s + 1408)r^7 \\
& + (-7452s^3 + 1248s^2 - 1584s - 576)r^6 \\
& + (-17496s^4 + 82944s^3 - 87936s^2 + 42240s - 5760)r^5 \\
& + (52488s^4 - 142884s^3 + 135144s^2 - 60144s + 8928)r^4 \\
& + (104976s^5 + 97200s^4 - 97056s^3 - 41472s^2 + 93824s - 26880)r^3 \\
& + (236196s^6 - 380538s^5 - 147744s^4 + 772080s^3 - 754944s^2 \\
& \quad + 333792s - 62208)r^2 \\
& + (236196s^6 - 734832s^5 + 1916784s^4 - 2658432s^3 + 2128320s^2 \\
& \quad - 919296s + 186624)r \\
& + 531441s^8 - 2204496s^7 + 5436882s^6 - 11805534s^5 + 17996446s^4 \\
& \quad - 17402304s^3 + 10364976s^2 - 3544992s + 557280.
\end{aligned}$$

3.1 No extraneous generic branch

Put $K = \mathbb{C}(s)$ and $C_0(t) = 2t^3 - 2t^2 + 2t - s$. The equation $s = 2t^3 - 2t^2 + 2t$ defines a degree-three rational function, so C_0 is irreducible over K and the intermediate field is $K(t)$. On $C_0 = 0$,

$$x(t) = \frac{2t^2}{3s + 2t^2 - 4t}, \quad y(t) = -\frac{3s + 2t^2 - 6t}{2t^2}.$$

The exact identities

$$\text{Res}_t(C_0, t) = s, \quad \text{Res}_t(C_0, 3s + 2t^2 - 4t) = 4s(27s^2 - 28s + 12)$$

show that the reconstruction denominators do not vanish generically.

Let $N(t, r, s)$ be the cleared numerator of the inner cubic and let R be its remainder modulo C_0 . Then $\deg_t R = 2$, and the checker verifies

$$\text{Res}_t(C_0, R) = 4s^7 P(r, s). \tag{6}$$

The subresultant degrees are 3, 2, 1, 0. Write the linear subresultant as

$$L(t, r, s) = \lambda(r, s)t + \mu(r, s).$$

Here $\deg_r \lambda = 6$, λ has 52 terms, and

$$\gcd(P, \lambda) = 1 \quad \text{in } \mathbb{C}(s)[r].$$

Thus, at the generic point of every component of $P = 0$, the common root is $t = -\mu/\lambda$; it satisfies $C_0 = R = 0$, and all reconstruction denominators are nonzero. Conversely, every genuine inverse pair annihilates the resultant. Hence the primitive resultant has no extraneous generic branch and has the same function field as the two-step inverse incidence curve.

4 Three inertia elements

4.1 A 9-cycle at infinity

Write $v = 1/s$ and consider $v^8 P(r, v^{-1})$. For the coefficients of $r^0, \dots, r^9$, their v -adic valuations are

$$0, 2, 2, 3, 4, 4, 5, 6, 8, 7.$$

Every intermediate point lies strictly above the line from $(0, 0)$ to $(9, 7)$. The lower Newton polygon is therefore one edge of slope $7/9$.

Set $v = w^9$ and $r = w^{-7}R$. The initial equation is

$$128R^9 + 531441 = 0.$$

Its roots are simple and lift uniquely to Puiseux branches. The local deck transformation $w \mapsto \zeta_9 w$ multiplies the leading term by ζ_9^{-7} . Since $\gcd(7, 9) = 1$, it permutes all nine branches in one orbit. Thus P is irreducible over $\mathbb{C}((1/s))$, hence over $\mathbb{C}(s)$, and slice monodromy contains a 9-cycle α .

Together with the linear-subresultant calculation, this gives the actual function-field tower

$$\mathbb{C}(s) \subset \mathbb{C}(s)(t) \subset \mathbb{C}(s)(t, r) = \mathbb{C}(s)(r), \quad [\mathbb{C}(s)(t) : \mathbb{C}(s)] = 3, \quad [\mathbb{C}(s)(r) : \mathbb{C}(s)] = 9.$$

In particular, the inner step has degree three and the nine Puiseux branches are precisely the nine sheets of the pulled-back $F \circ F$ cover.

4.2 A single transposition

Exact discriminant factorization gives

$$\text{disc}_r(P) = 2^{38} q(s)^8 A(s) B(s)^2, \quad q(s) = 27s^2 - 28s + 12, \quad (7)$$

where

$$\begin{aligned} A(s) = & 1129718145924s^{12} - 6474958632657s^{11} + 21955119111630s^{10} \\ & - 60661410535386s^9 + 123515279853390s^8 - 191093865073182s^7 \\ & + 235490291194248s^6 - 188631015819696s^5 + 54561577345568s^4 \\ & + 41882989175328s^3 - 44714997684096s^2 + 15094447332352s \\ & - 1434819245056. \end{aligned}$$

The exact certificates verify that A is squarefree and coprime to q , B , and the leading coefficient $128s$. At each root of A , the discriminant has valuation one while the degree remains nine. Local geometric inertia is thus a single transposition β . A permutation moving only two sheets cannot act nontrivially on the three blocks, so β is supported in one block.

The full degree-22 factor B is recorded in Appendix A and checked independently in SymPy and PARI/GP.

4.3 A transposition of the blocks

The outer cubic has discriminant

$$\text{disc}_t(2t^3 - 2t^2 + 2t - s) = -4(27s^2 - 28s + 12) = -4q(s).$$

The quadratic q is squarefree and coprime to A and B . A small loop about either root of q has outer monodromy a transposition. Total inertia contains an element γ whose image on the three blocks is that transposition.

5 The group argument

Lemma 4. *Let $H \leq S_3 \wr S_3$ in its natural action on three blocks of size three. Suppose H contains a 9-cycle α , a single transposition β , and an element γ inducing a transposition on the blocks. Then $H = S_3 \wr S_3$.*

Proof. The 9-cycle induces a 3-cycle on the blocks, and α^3 acts as a 3-cycle inside each block. The transposition β is supported in one block. Conjugating it by α^3 gives a distinct transposition in that block, so these elements generate the full S_3 supported there. Conjugation by powers of α gives the full S_3 independently in all three blocks. Hence the base subgroup S_3^3 lies in H .

On blocks, α supplies a 3-cycle and γ a transposition, so the image of H is the top S_3 . A subgroup containing the full base and surjecting onto the top is the full semidirect product. $\square$

The three inertia elements constructed above satisfy the lemma, so the slice monodromy is $S_3 \wr S_3$. Slice monodromy is a subgroup of generic monodromy; combining this with (5) proves the level-two assertion of Theorem 1.

6 Full-cycle inertia for all iterates

We prove Proposition 2 by a local-field induction. Put $u = 1/s$, let $K = \mathbb{C}((u))$ with $\nu(u) = 1$, and work in a fixed Puiseux closure. Start from $X_0 = (1, 2, u^{-1})$ and choose a compatible inverse tower $F(X_{k+1}) = X_k$. We prove that

$$\nu(x_k) = a_k, \quad \nu(y_k) = -a_k, \quad \nu(z_k) = -c_k, \quad (8)$$

with nonzero leading units, and that the level- k local branch field has ramification degree 3^k over K . Initially $a_0 = 0$ and $c_0 = 1$.

The next inverse parameter satisfies

$$2x_k t^3 - y_k t^2 + 2t - z_k = 0.$$

The coefficient valuations, in increasing powers of t , are

$$(-c_k, 0, -a_k, a_k).$$

If $c_k > 5a_k$, the two middle points lie strictly above the segment joining $(0, -c_k)$ to $(3, a_k)$. The Newton polygon theorem gives, for all three roots,

$$\nu(t_{k+1}) = -\frac{a_k + c_k}{3}. \quad (9)$$

The endpoint residual polynomial is a separable binomial cubic.

We now check every possible cancellation in the reconstruction formulas. In

$$y_{k+1} = -\frac{y_k t^2 + 3z_k - 6t}{2t^2},$$

the term $3z_k$ is uniquely smallest, since

$$-c_k < -a_k + 2\nu(t), \quad -c_k < \nu(t).$$

Consequently $\nu(y_{k+1}) = (2a_k - c_k)/3 = -a_{k+1}$. Next, $\nu(ty_{k+1}) = (a_k - 2c_k)/3 < 0$, so ty_{k+1} uniquely dominates 1 in $x_{k+1} = t/(1 - ty_{k+1})$ and $\nu(x_{k+1}) = a_{k+1}$. Finally, $-z_k$ uniquely dominates the numerator of

$$z_{k+1} = \frac{2x_{k+1} - 3x_{k+1}^2 y_{k+1} - z_k}{x_{k+1}^3}.$$

Thus (8) persists with

$$a_{k+1} = \frac{c_k - 2a_k}{3}, \quad c_{k+1} = 2(c_k - a_k). \quad (10)$$

For $\rho_k = a_k/c_k$, the interval $0 \leq \rho_k \leq 1/6$ is invariant under

$$\rho_{k+1} = \frac{1 - 2\rho_k}{6(1 - \rho_k)},$$

and its image is contained in $[2/15, 1/6]$ after the first step. Hence $c_k > 5a_k$ at every level, completing the valuation induction.

For ramification, set $A_k = 3^k a_k$ and $C_k = 3^k c_k$. Then

$$A_{k+1} = C_k - 2A_k, \quad C_{k+1} = 6(C_k - A_k), \quad A_0 = 0, \quad C_0 = 1.$$

At the m th inverse step,

$$\nu(t_m) = -\frac{E_m}{3^m}, \quad E_m = A_{m-1} + C_{m-1}.$$

For $m \geq 2$, $3 \mid C_{m-1}$ and $A_{m-1} \equiv 1 \pmod{3}$; also $E_1 = 1$. Thus $3 \nmid E_m$ for every m . By induction the preceding branch field has value group $(1/3^{m-1})\mathbb{Z}$, whereas $\nu(t_m)$ has exact denominator 3^m . Adjoining t_m therefore has ramification degree at least three; its equation is cubic, so both relative degree and ramification degree equal three. Reconstruction is rational and $t_m = x_m/(1 + x_m y_m)$, hence passing between t_m and X_m loses no field.

The level- m local branch consequently has degree and ramification index 3^m over K . The global slice cover has degree at most 3^m , so this branch exhausts it and is totally ramified. Newton–Puisseux theory in characteristic zero makes tame inertia cyclic and transitive on the 3^m conjugates; a generator is a single 3^m -cycle.

As an independent check, nested PARI resultants for $m = 3$ give a degree-27 eliminant with one lower Newton edge from $(0, 0)$ to $(27, 34)$.

7 Full wreath monodromy at level three

We retain the discriminant information from the same exact eliminant. On the target line $(1, 2, s)$, let

$$C_0(t) = 2t^3 - 2t^2 + 2t - s, \quad X_1 = \text{rec}((1, 2, s), t),$$

where rec denotes (4). Let $C_1(t, r)$ be the numerator of $C_{X_1}(r)$, set $X_2 = \text{rec}(X_1, r)$, and let $C_2(t, r, q)$ be the numerator of $C_{X_2}(q)$. Define

$$\mathcal{R}(q, s) = \text{Res}_t(C_0, \text{Res}_r(C_1, C_2)), \quad Q = \frac{\mathcal{R}}{\text{content}_q(\mathcal{R})}.$$

Exact PARI arithmetic gives

	$\deg_q \mathcal{R}$	$\deg_s \mathcal{R}$	$\deg_s \text{content}_q(\mathcal{R})$	$\deg_q Q$	$\deg_s Q$
value	27	244	196	27	48.

After $u = 1/s$ and the common valuation shift, the lower Newton polygon is the single edge $(0, 0) - (27, 34)$. Since $\gcd(27, 34) = 1$, each irreducible factor over $\mathbb{C}((u))$ would have degree divisible by 27. Thus Q is irreducible over $\mathbb{C}((u))$, hence over $\mathbb{C}(s)$, and inertia contains a 27-cycle α .

There is no hidden denominator branch. If D_1, D_2 are the denominators cleared when defining C_1, C_2 , respectively, the checker forms

$$L(s) = \text{Res}_t(C_0, D_1), \quad U(s) = \text{Res}_t(C_0, \text{Res}_r(C_1, D_2)).$$

Both are nonzero. Choose a generic inverse tower and its deepest resolvent parameter q . Reconstruction is defined there, and elimination gives $Q(q, s) = 0$. Since Q is irreducible of degree 27, it is the minimal polynomial of this genuine q . Thus the inverse-tower field has degree at least 27, while the three cubic steps give degree at most 3^3 . Equality holds, q generates the tower field, and Q is exactly the generic fiber polynomial of the pulled-back third iterate.

Let $\mathcal{D} = \text{disc}_q(Q)$ and put

$$G = \gcd(\mathcal{D}, \mathcal{D}'), \quad R_{\text{sf}} = \mathcal{D}/G, \quad M = \gcd(R_{\text{sf}}, G), \quad E = R_{\text{sf}}/M.$$

Thus E is the product of precisely the irreducible discriminant factors of multiplicity one. The exact degrees are

$$\deg_s \mathcal{D} = 1612, \quad \deg_s R_{\text{sf}} = 752, \quad \deg_s M = 676, \quad \deg_s E = 76.$$

Furthermore, E is squarefree and coprime to M , to the degree-27 leading coefficient of Q (of s -degree 14), and to L and U . At every root of E , the degree therefore remains 27, reconstruction is genuine, and the discriminant has valuation one. Local geometric inertia is a single transposition τ . Since monodromy preserves the nine bottom blocks of size three, τ lies inside one bottom block.

Lemma 5. *Let $H \leq W_3$ contain a 27-cycle and a single transposition. If the image of H on the nine bottom blocks is W_2 , then $H = W_3$.*

Proof. The 27-cycle α induces a 9-cycle on the bottom blocks, and α^9 acts as a 3-cycle in each such block. The elements τ and $\alpha^9 \tau \alpha^{-9}$ generate the full S_3 on one bottom block. Conjugation by powers of α supplies independent S_3 factors on all nine bottom blocks, so $S_3^9 \leq H$. This is the complete kernel of $W_3 \rightarrow W_2$. Since H maps onto W_2 , it follows that $H = W_3$. $\square$

The image of $\text{Mon}(F^{\circ 3})$ on the bottom blocks is $\text{Mon}(F^{\circ 2}) = W_2$ by the level-two argument in Sections 3–5. Applying Lemma 5 gives

$$\text{Mon}(F^{\circ 3}) = W_3, \quad |W_3| = 6^{9+3+1} = 13,060,694,016.$$

This paper proves level three only. The accompanying bounded-memory level-four certificate was subsequently computed and independently audited. It supplies the required deepest-block transposition and proves $\text{Mon}(F^{\circ 4}) = W_4$. Its localization and norm-to-inertia argument remain outside this level-three paper.

8 Independent arithmetic cross-check

At $s = -3$, the primitive specialization is

$$\begin{aligned} f(r) = & -384r^9 - 256r^8 + 33952r^7 + 216612r^6 - 4580568r^5 \\ & + 9515052r^4 - 15697056r^3 + 223986114r^2 + 599887620r \\ & + 17172300267. \end{aligned}$$

Dependency-free modular arithmetic verifies squarefree factorizations with irreducible-degree patterns

p	degrees
13	(9)
61	(2, 1, 1, 1, 1, 1, 1, 1)
19	(2, 2, 2, 2, 1).

These give Frobenius cycle types (9), $(2, 1^7)$, and $(2^4, 1)$. The same group lemma forces the arithmetic group to be the full wreath product. GAP independently enumerates all subgroups of the wreath product and finds no proper survivor. PARI/GP 2.17.4 identifies five unrelated rational fibers as transitive group 9T31 of order 1296.

This is corroboration only; neither geometric proof infers geometric monodromy from an arithmetic specialization.

9 Verification and priority scope

The artifact contains complementary exact checks:

File	Scope
<code>verify_symbolic.py</code>	Jacobian, collision, inverse identities, resultant, denominator and subresultant certificates, Newton data, discriminant, squarefreeness, coprimality
<code>verify_modular.py</code>	dependency-free polynomial arithmetic and Rabin irreducibility certificates
<code>verify_iterate_inertia.py</code>	dependency-free recurrence, Newton-edge, and reconstruction-dominance checks
<code>verify_pari.gp</code>	independent discriminant, coprimality, finite-field, and arithmetic Galois checks
<code>verify_group.g</code>	exhaustive subgroup check inside $S_3 \wr S_3$
<code>verify_level3_newton.py</code>	independent degree-27 resultant and Newton edge for the third iterate
<code>verify_level3_wreath.py</code>	primitive degree-27 eliminant, simple discriminant divisor, leading and denominator guards, and depth-three group lemma

The level-three checkers use SymPy-generated equations and delegate nested resultants and discriminants to PARI.

The closest public monodromy study states only $\text{Mon}(F_0 \circ F_0) \leq S_3 \wr S_3$ and calls its exact group a natural next computation [2]. A separate four-variable construction already claims a different imprimitive degree-eight group of order 192 [3]. We therefore do not claim the first imprimitive or first non-symmetric Keller monodromy. The claimed contribution is the exact second- and third-iterate groups, the realizations $(9, W_2)$ and $(27, W_3)$, and the all-iterate full-cycle statement. A timestamped 23-repository audit accompanies the artifact.

The stronger arboreal assertion that every iterate has the full iterated wreath product remains open: the present paper does not produce a new single transposition at every level. The separate certificate in `w4_search/RESULT.md` settles W_4 .

A The degree-22 square factor

$$\begin{aligned} B(s) = & 1423119505038213888s^{22} - 155058052818404824443s^{21} + 1413754646027656083066s^{20} \\ & - 8145658955220494785812s^{19} + 33677018812011807334224s^{18} - 108385682498649366622416s^{17} \\ & + 280859820926917245978240s^{16} - 598631883156564470992728s^{15} + 1062288637590844067815584s^{14} \\ & - 1579910926841521168581900s^{13} + 1974730823883289142626824s^{12} - 2073348574060015592229024s^{11} \\ & + 1822485349587628391880960s^{10} - 1333523184375693801555072s^9 + 806144943610022071172864s^8 \\ & - 398941760796329492797440s^7 + 159756878618286560778240s^6 - 50952869532774134959104s^5 \\ & + 12636860641575849510912s^4 - 2344566242453066735616s^3 + 304677812001210531840s^2 \\ & - 24551525690267664384s + 926711257485017088. \end{aligned}$$

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